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ABSTRACT

This is unit three of a fifteen-unit secondary mathematics textbook. This unit contains two chapters. The first chapter discusses integers and the second chapter discusses rational numbers. Operations with both types of numbers as well as the structure of the systems are discussed. (MK)

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**SCHOOL
MATHEMATICS
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UNIT THREE

Chapter 5. The Integers and Their Structure

Chapter 6. Rational Numbers

SE 031 008



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Chapter 5

THE INTEGERS AND THEIR STRUCTURE

5-1. Introduction

In the last chapter we learned that the world of mathematics and the world of reality (the real world) can be connected in terms of mathematical models. Furthermore, it is the mathematical model which enables us to find answers to so many of the problems we meet in our complicated world. In this chapter we are going to use a familiar model, the number line, to help us organize and expand our knowledge of number. The number line is an interesting model because it uses one idea (the idea of line) to model another idea (the idea of number).

5-2. The Number Line

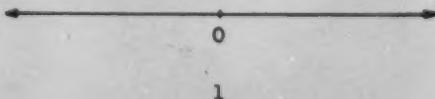
A number is an idea, and the development of a good number system took hundreds of years. To help understand numbers and their uses, many schemes have been used. One of the most successful ways to learn about numbers and their behavior is to make use of the number line. Let's recall the construction of the number line as a starting point for further discussion of numbers.

We begin by considering the set of whole numbers, $\{0, 1, 2, 3, 4, \dots\}$, together with a line in space. Our purpose is to associate each number belonging to the set of whole numbers with exactly one point belonging to the line. While there are many ways to do this, we select an association such that the constructed number line will be useful to us in our investigation of number and number behavior.

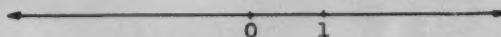
Below is a drawing of a horizontal line in space.



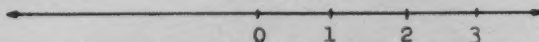
First, we associate the number 0 with any point of the line. Remember that we call this point the origin.



Next, we associate the number 1 with another point of the line, usually a point to the right of the origin.



After this, we have no freedom of choice in determining points to correspond to the other whole numbers.



Notice that this correspondence from whole numbers to points is a function.

Although it is possible to associate a number with each point of the line, we are not going to do so in this chapter. Our attention, at this time, will concern some of these points:

- (1) the points already corresponding to whole numbers, and
- (2) certain points, to the left of the origin, which we soon will correspond to numbers closely related to the whole numbers.

In the remaining part of this chapter, as well as in all following chapters, the set of whole numbers will be called the set of non-negative integers. If the number 0 is removed from this set, the new set of numbers, $\{1, 2, 3, 4, \dots\}$, is called the set of positive integers. You probably have been referring to the positive integers as the counting numbers. We prefer the name positive integers.

For convenience, we will often write about a point as if it represented a number. But, of course, we will always mean that the point corresponds to the number.

Exercises 5-2a

(Class Discussion)

1. Why do you think a horizontal line was selected in our construction of the number line, and why do you think the number 1 was selected to be to the right of the number 0?
2. In the set $\{0, 1, 2, 3, \dots\}$ what do the three dots indicate?
3. What is the smallest positive integer represented on the number line?
4. What is the largest positive integer represented on the number line?

5. What is the smallest non-negative integer represented on the number line?
6. If we choose a point on the number line which represents a non-negative integer, where would we find a point for a non-negative integer which is greater than the first integer?
7. If one non-negative integer is less than another, what do you know about their respective positions on the number line?
8. When the number 0 is included in the set of positive integers, why is the new set of numbers called the set of non-negative integers?
9. In our construction of the number line we stated that after we had chosen points to represent the number 0 and the number 1, we no longer had freedom of choice in completing this construction. Explain what is meant by this statement.

Now, we are ready to associate numbers with certain points on the number line which are to the left of the origin. Suppose we imagine the pivot point of a compass placed at the origin and the pencil point placed at the point corresponding to 1 (see Figure 1). Now, imagine that the pencil point of this compass makes a half circle.

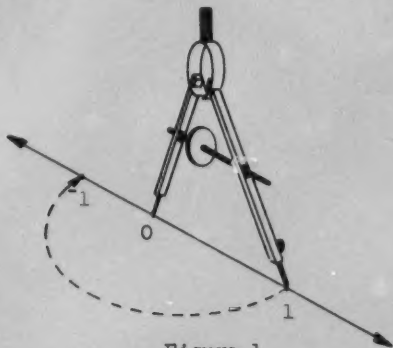


Figure 1

In this way, we have located a point to the left of the origin that is the same distance from zero as the point corresponding to 1. The point to the left of zero corresponds to a number which we will name negative one, written -1. In a like manner we can locate other points on the half-line to the left of the origin and, then, associate numbers with them (see Figure 2).

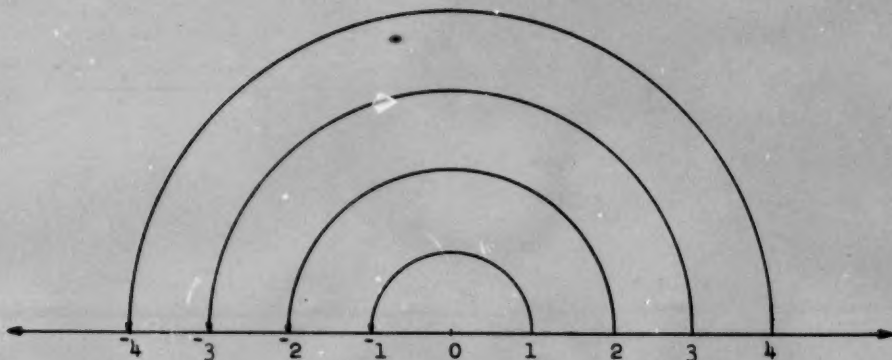


Figure 2

Since the numbers we associated with points on the right half-line were named positive integers, it seems only natural that these new, but closely related, numbers, $\{\dots, -4, -3, -2, -1\}$, be named negative integers. When we unite the negative integers, zero, and the positive integers, $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, we call this new set of numbers the set of integers.

Any one of the numbers in the set is called an integer. For example, 8 is an integer, -43 is an integer, and so on. (On the other hand, $\frac{1}{2}$ is not an integer.)

Exercises 5-2b

(Class Discussion)

1. How does our development of the integers by use of the number line relate to the idea of separation that you studied in Chapter 1?
2. Are there any other integers beside the positive and negative integers?
3. When we use the horizontal number line, we know that:
 - (1) one non-negative integer is greater than ($>$) another if it is to the right of the other on the number line, and
 - (2) one non-negative integer is less than ($<$) another if it is to the left of the other on the number line.

Let's agree to use the thinking expressed in statements (1) and (2) above, to determine the order of any two different integers on the number line. For example, study the pairs of true sentences shown below:

- | | | |
|---------------|--------------|--------------|
| (a) $-4 < -3$ | (b) $-2 < 0$ | (c) $-1 < 6$ |
| $-3 > -4$ | $0 > -2$ | $6 > -1$ |

Now, determine which of the following sentences are true and which are false.

- | | | | |
|---------------|---------------|----------------|---------------|
| (a) $0 < 71$ | (c) $-7 < 4$ | (e) $7 > 2$ | (g) $-9 > 3$ |
| (b) $-71 < 0$ | (d) $-1 < -5$ | (f) $-4 > -15$ | (h) $-6 > -2$ |

4. What is the largest integer represented on the number line?
5. What is the smallest integer represented on the number line?

5-3. The Opposite Function

When we consider the set of integers, as pictured in Figure 2, we see that for each positive integer there is a related negative integer. These two integers are:

- (1) on opposite sides of the origin, and
- (2) the same distance from the origin.

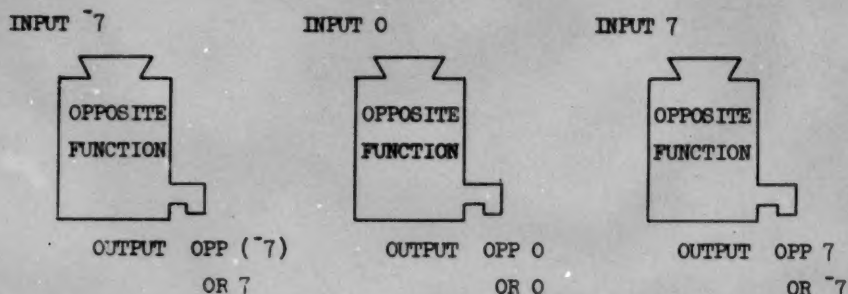
When numbers are located in this way, we say that either number is the opposite of the other. Therefore, the opposite of 1 is -1, and the opposite of -1 is 1. In the same manner, the opposite of -5 is 5, and the opposite of 5 is -5.

Although it would be foolish to consider an opposite for zero in terms of statement (1), above, we will find it useful to agree that the opposite of zero is itself, that is, the opposite of 0 is 0.

Since corresponding to each integer there is exactly one integer, its opposite, a function is suggested. We will call this function the opposite function, and the symbol, opp (meaning "the opposite of"), will be placed to the left of a numeral to indicate the output for the function. Therefore, the opposite function can be described by

$$f : x \rightarrow \text{opp } x.$$

If we think of the opposite function as a machine, it would work as pictured below.



In function notation:

$$-7 \rightarrow 7$$

$$0 \rightarrow 0$$

$$7 \rightarrow -7$$

In general:

- (1) If x represents a positive number ($x > 0$), the output for the opposite function is a negative number, that is, $\text{opp } x < 0$.
- (2) If x represents a negative number ($x < 0$), the output for the opposite function is a positive number, that is, $\text{opp } x > 0$.
- (3) If x represents zero ($x = 0$), the output for the opposite function is zero, that is, $\text{opp } 0 = 0$.

Exercises 5-3

1. Find common names for each of the following:

(a) $\text{opp } 5$	(d) $\text{opp } (3 + 5)$
(b) $\text{opp } (-8)$	(e) $\text{opp } (3 \cdot 5)$
(c) $\text{opp } 0$	(f) $\text{opp } (7 - 4)$
2. Below are the coordinates of points in the coordinate plane. Plot, and name, every point on one coordinate plane.

(a) $A(-4, \text{opp } -2)$	(c) $C(4, \text{opp } 2)$
(b) $B(\text{opp } -4, 2)$	(d) $D(\text{opp } 4, -2)$

3. Concerning Exercise 2, use a ruler to connect point A to point B, point B to point C, point C to point D, and point D to point A.
- What kind of figure is formed?
 - Connect point A to point C and point B to point D. What seem to be the coordinates of the point of intersection of $\overline{AC}$ and $\overline{BD}$?
4. Find each of the following:
- the opposite of opp 3.
 - the output under the opposite function for an input of opp (-2).
 - the opposite of opp (-5).
 - opp opp 3.
 - opp opp (-3).
5. (a) Complete each of the tables shown below.

Table 1

Input: x	-3	-2	-1	0	1	2	3
Output: x							

Table 2

Input: x	-3	-2	-1	0	1	2	3
Output: opp x							

- Name the function which is partly represented by Table 1.
 - Name the function which is partly represented by Table 2.
 - On one coordinate plane, draw the graphs of these functions.
 - Use the language of geometry to describe accurately these graphs.
-

5-4. The Absolute Value Function

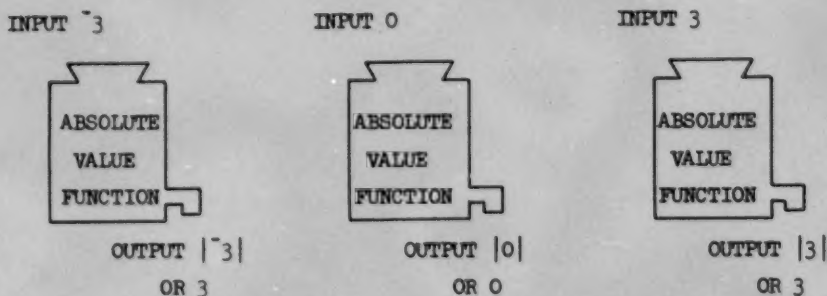
On the number line, we know that 3 names an integer which is 3 units to the right of zero, and that -3 names an integer which is 3 units to the left of zero. In other words, we know that the name of an integer indicates its direction, as well as its distance, from zero.

In mathematics we often need to indicate the distance between some number and zero on the number line without any concern for its direction. When this is the case we refer to this distance as the absolute value of the number. Therefore, the absolute value of 3 is 3, and the absolute value of -3 is 3. Of course, we are saying that the distance between three and zero is 3 units, and that the distance between negative three and zero is 3 units.

We can see that for each integer there is a related undirected distance from zero. Here again, we have a correspondence which is a function. That is, corresponding to each integer there is exactly one undirected distance from zero, the absolute value of the integer. We will call this function the absolute value function, and vertical bars, || (meaning "the absolute value of"), will be placed on both sides of a numeral to indicate the output for the function. Therefore, the absolute value function can be described by

$$f : x \rightarrow |x|.$$

If we think of the absolute value function as a machine, it would work as pictured below.



In function notation:

$$-3 \rightarrow 3$$

$$0 \rightarrow 0$$

$$3 \rightarrow 3$$

In general:

- (1) If x represents a non-negative number ($x \geq 0$), the output for the absolute value function is x , that is, $|x| = x$.
- (2) If x represents a negative number ($x < 0$), the output for the absolute value function is $\text{opp } x$, that is, $|x| = \text{opp } x$.

Note: The absolute value of a number can never be a negative number, that is, $|x| \geq 0$.

Exercises 5-4

1. Find common names for each of the following:
(a) $|-8|$ (d) $\text{opp } |2|$
(b) $|7|$ (e) $\text{opp } |-2|$
(c) $|0|$ (f) $\text{opp opp } |\text{opp } 3|$
2. Translate each numeral, below, into an English phrase.
(a) $|\text{opp } 4|$
(b) $\text{opp } |4|$
3. The common name for $|\text{opp } 4|$ can be determined as follows:

$$|\text{opp } 4| = |-4| = 4.$$

Now, find the common name for $\text{opp } |4|$.

4. Explain why the equation, below, is true when x is replaced by any integer.

$$|x| = |\text{opp } x|$$

5. (a) Complete the table shown below.

Input:	x	-5	-4	-3	-2	-1	0	1	2	3	4	5
Output:	$ x $											

- (b) Name the function which is partly represented by this table.
- (c) Draw the graph of the function.

- (d) Use the language of geometry to describe accurately this graph.
- (e) How is the graph of this function related to the graphs of the identity function and the opposite function?
-

5-5. Integers and Arrows

At this time, we are going to learn how to represent any non-zero integer on the number line by the use of an arrow.

As a starting point, let's agree to a way of representing any positive integer, such as 3. We will represent this integer by an arrow that has the following characteristics:

- (1) a length of $|3|$ units, and
- (2) a direction to the right.

For example, either of the arrows (see Figure 3) can be used to represent 3 (positive three).

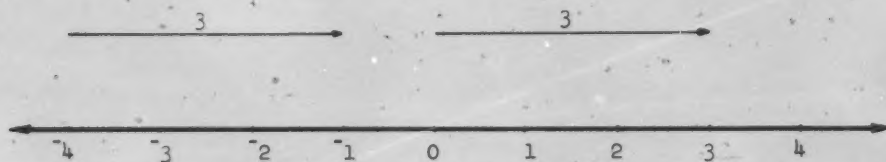


Figure 3

Now, we can represent any negative integer, such as -2 . We can represent this integer by an arrow that has the following characteristics:

- (1) a length of $|-2|$ units, and
- (2) a direction to the left.

For example, either of the arrows (see Figure 4) can be used to represent -2 (negative two).

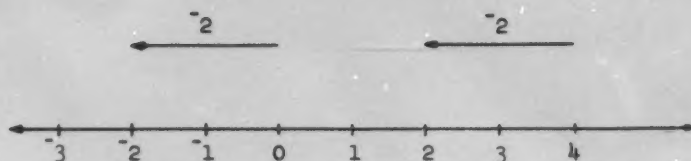


Figure 4

In Figure 3 and Figure 4, we have drawn the arrows directly above the number line instead of on it so that they can be clearly seen.

5-6. Addition of Integers

In the last section we learned that we can represent a non-zero integer on the number line by means of an arrow. For a particular non-zero integer, we agreed that:

- (1) the length of the arrow will correspond to the absolute value of the integer,
- (2) the arrow will point right if the integer is positive,
- (3) the arrow will point left if the integer is negative.

Now, with these three simple ideas, we will learn how to add any two integers.

Consider the addition problem $4 + \text{opp } 3$. We are instructed to add the opposite of 3 to 4. To do this, let's think of our first addend, 4, as a "starting point" for this operation (see Figure 5).

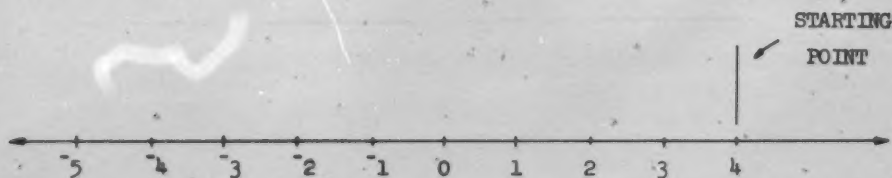
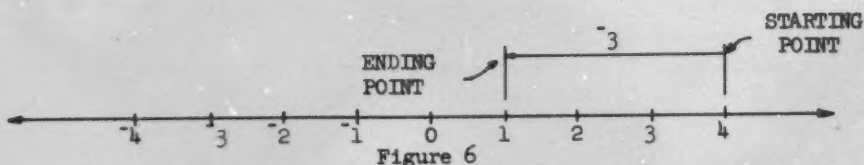


Figure 5

Now, let's place an arrow, of length 3 units, so that it starts at 4 and points to the left (see Figure 6).



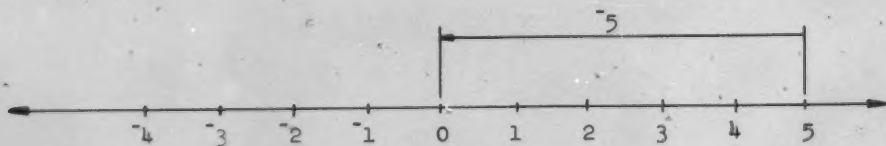
We end at the point whose coordinate is 1. Therefore, $4 + \text{opp } 3 = 1$.

To summarize, to add any two integers, $a + b$, on the number line:

- (1) locate the point whose coordinate is a, and
- (2) move right or left (depending on whether b is positive or negative) |b| units.

The coordinate of the point at which you end is the sum a + b.

One problem of special interest in addition of integers is the sum of an integer and its opposite. Consider the problem $5 + \text{opp } 5$. If we follow the procedure described for addition of integers, our picture will be as follows.



We can see that we end at the point whose coordinate is zero. Our intuition tells us that the sum of any integer and its opposite will always be zero.

Earlier, we wrote that when two integers are located on opposite sides of zero and at the same distance from zero (have the same absolute value), then either number is called the opposite of the other. Now, we have learned that if we add two integers that are related in this way, their sum will be zero. This enables us to define opposites in another fashion. In general, if a and b represent integers such that

$$a + b = 0,$$

then b is the opposite of a and, also, a is the opposite of b.

Exercises 5-6a

(Class Discussion)

1. Given two integers, a and b, answer the following questions concerning their sum. A sketch on the number line will help you to answer each question.

- (a) If a is a positive integer ($a > 0$) and b is a positive integer ($b > 0$), then the sum ($a + b$) will be a _____ ? integer.
- (b) If a is a negative integer ($a < 0$) and b is a negative integer ($b < 0$), then the sum ($a + b$) will be a _____ ? integer.
- (c) If a is a positive integer and b is a negative integer and the absolute value of a is greater than the absolute value of b ($|a| > |b|$), then the sum ($a + b$) will be a _____ ? integer.
- (d) If a is a positive integer and b is a negative integer and the absolute value of a is less than the absolute value of b ($|a| < |b|$), then the sum ($a + b$) will be a _____ ? integer.
- (e) If a is a negative integer and b is a positive integer and the absolute value of a is greater than the absolute value of b ($|a| > |b|$), then the sum ($a + b$) will be a _____ ? integer.
- (f) If a is a negative integer and b is a positive integer and the absolute value of a is less than the absolute value of b ($|a| < |b|$), then the sum ($a + b$) will be a _____ ? integer.
- (g) If a is a positive integer and b is the opposite of a ($b = \text{opp } a$), then b is a _____ ? integer.
- (h) If a is a positive integer and b is the opposite of a ($b = \text{opp } a$), then the sum ($a + b$) will be the integer _____ ? .

- (i) If a is a negative integer and b is the opposite of a ($b = \text{opp } a$), then b is a _____ integer.
- (j) If a is a negative integer and b is the opposite of a ($b = \text{opp } a$), then the sum $(a + b)$ will be the integer _____.

Exercises 5-6b

1. Find the sum for each of the problems (a) through (j).
- | | | |
|-------------------|------------------|------------------|
| (a) $2 + 7$ | (e) $(-8) + 11$ | (i) $8 + (-8)$ |
| (b) $(-3) + (-1)$ | (f) $(-15) + 10$ | (j) $(-14) + 14$ |
| (c) $9 + (-5)$ | (g) $0 + 4$ | |
| (d) $10 + (-12)$ | (h) $(-2) + 0$ | |
- (k) We know that the sum of two positive integers is always a positive integer (The Closure Property of Addition). Do you think that the sum of any two integers will always be an integer? If not, find a counterexample (a case where this is not true).
2. Find the sum for each of the problems (a) through (l).
- | | | |
|-------------------|----------------|----------------|
| (a) $5 + 9$ | (e) $6 + (-2)$ | (i) $(-5) + 0$ |
| (b) $9 + 5$ | (f) $(-2) + 6$ | (j) $0 + (-5)$ |
| (c) $(-4) + (-6)$ | (g) $(-9) + 7$ | (k) $1 + (-1)$ |
| (d) $(-6) + (-4)$ | (h) $7 + (-9)$ | (l) $(-1) + 1$ |
- (m) We know that when we add two positive integers the order in which we add them will not change their sum (The Commutative Property of Addition). Do you think that any two integers will behave in this way? If not, find a counterexample.

3. Find the sum for each of the problems (a) through (h). Perform the addition within the parentheses first.

(a) $(33 + 25) + 75$

(e) $(9 + 13) + (-13)$

(b) $33 + (25 + 75)$

(f) $9 + (13 + (-13))$

(c) $((-5) + (-5)) + (-8)$

(g) $((-3) + (-7)) + 10$

(d) $(-5) + ((-5) + (-8))$

(h) $(-3) + ((-7) + 10)$

- (i) We know that when we add three positive integers the manner in which we group the middle addend will not change the sum (The Associative Property of Addition). Do you think that any three integers will behave in this way? If not, find a counterexample.
4. We know that the sum of any non-negative integer, a, and zero is a (The Addition Property of Zero). Do you think that zero will behave this way with any integer? If not, find a counterexample.
5. Find the sum for each of the problems (a) through (f).

(a) $5 + (-5)$

(c) $\text{opp } 238 + 238$

(b) $(-17) + 17$

(d) $(-1) + \text{opp } (-1)$

- (e) Five sets are shown below. Identify the set whose members have opposites that are also members of the set.

(1) $\{1, 2, 3, 4, \dots\}$

(4) $\{\dots, -4, -3, -2, -1, 0\}$

(2) $\{0, 1, 2, 3, 4, \dots\}$

(5) $\{\dots, -2, -1, 0, 1, 2, \dots\}$

(3) $\{\dots, -3, -2, -1\}$

6. Find the unknown addend in each of the following sentences. A sketch on the number line will help you to answer each question.

(a) $2 + \underline{\quad} = 6$

(g) $-5 + \underline{\quad} = 0$

(b) $2 + \underline{\quad} = 2$

(h) $-5 + \underline{\quad} = -13$

(c) $2 + \underline{\quad} = 0$

(i) $0 + \underline{\quad} = 12$

(d) $-2 + \underline{\quad} = -2$

(j) $0 + \underline{\quad} = 0$

(e) $-5 + \underline{\quad} = 3$

(k) $0 + \underline{\quad} = -12$

(f) $-5 + \underline{\quad} = -5$

7. If two integers, a and b , are added what do we know about them if their sum is:

- (a) negative?
- (b) zero?
- (c) positive?

5-7. A Remarkable Discovery

Suppose you buy an item which costs 9 cents, and give the cashier 25 cents. When the cashier gives you your change, what does she do? She puts down one cent and says "10", one nickel and says "15", and one dime and says "25". What has she been doing? She has been subtracting 9 from 25.

$$25 - 9 = x$$

How does she do it? -- by finding what she has to add to 9 in order to get 25.

$$9 + x = 25$$

Of course, the equation " $25 - 9 = x$ " (indicating a subtraction problem) means the same thing as the equation " $9 + x = 25$ " (indicating an addition problem). Furthermore, our knowledge of arithmetic tells us that either sentence will be true when x is replaced by 16. Thus, $25 - 9 = \underline{16}$ and $9 + \underline{16} = 25$.

The problem, above, focused our attention on three different numbers: 25, 9, and 16. Of the three, the "starting" number 25, and the "ending" number 16 seem to be the important ones. Let's draw the number line and locate the points whose coordinates are 25 and 16 (see Figure 7).

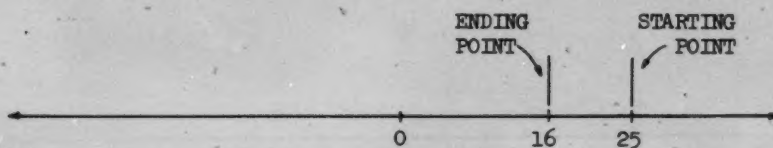
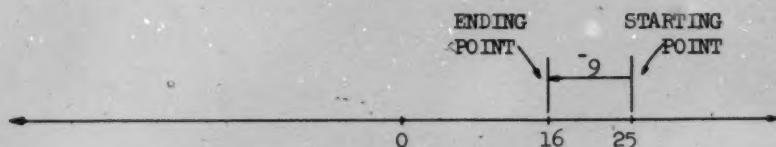


Figure 7

20

We don't have to inspect Figure 7 for long to realize that we have seen something like this before (see Figures 5 and 6 in this chapter)!

Now, if we place the tail of an arrow at the starting point and the head of this arrow at the ending point, we can determine that this arrow is a representation of the integer opp 9.



The number line, above, pictures the addition problem,

$$25 + \text{opp } 9 = 16.$$

But, we already know that,

$$25 - 9 = 16.$$

Therefore,

$$25 - 9 = 25 + \text{opp } 9.$$

The sentence, $25 - 9 = 25 + \text{opp } 9$, suggests that the subtraction of one integer from another can be expressed in terms of addition!

In fact, since for each integer there is an integer which is its opposite, we can say:

If a and b represent any two integers, then

$$a - b = a + \text{opp } b!$$

Study these examples:

$$8 - 8 = 8 + \text{opp } 8 = 0$$

$$(-2) - (-2) = (-2) + \text{opp } (-2) = 0$$

$$5 - 0 = 5 + \text{opp } 0 = 5$$

$$(-5) - 0 = (-5) + \text{opp } 0 = -5$$

$$7 - 4 = 7 + \text{opp } 4 = 3$$

$$4 - 7 = 4 + \text{opp } 7 = -3$$

$$8 - (-1) = 8 + \text{opp } (-1) = 9$$

$$(-1) - 8 = (-1) + \text{opp } 8 = -9$$

$$(-3) - 5 = (-3) + \text{opp } 5 = -8$$

$$5 - (-3) = 5 + \text{opp } (-3) = 8$$

$$(-2) - (-3) = (-2) + \text{opp } (-3) = 1$$

$$(-3) - (-2) = (-3) + \text{opp } (-2) = -1$$

Exercises 5-7

1. Supply the unknown addend in each sentence.

(a) $8 + 5 + (?) = 8$

(b) $6 + (-3) + (?) = 6$

(c) $(-11) + 6 + (?) = -11$

(d) $(-11) + (-6) + (?) = -11$

(e) $(-3) + (?) + (-8) = (-8)$

(f) $(-3) + (-7) + (?) = -7$

(g) $1 + (-1) + (?) = 1$

(h) $1 + (?) + 1 = 1$

2. For each sentence, find the replacement for x that makes the sentence true.

(a) $(-5) + 2 = x$

(c) $8 + x = -3$

(b) $(-3) + x = 8$

(d) $x + (-4) = 11$

3. Perform the following subtractions by expressing each problem in terms of addition. For example, $(-4) - 2 = (-4) + (-2) = -6$.

(a) $(-4) - 2$

(e) $(-5) - 1$

(b) $(-6) - (-1)$

(f) $(-3) - (-7)$

(c) $8 - (-3)$

(g) $35 - (-2)$

(d) $(-11) - (-13)$

(h) $75 - (-6)$

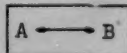
4. Perform the following subtractions:

- | | |
|----------------------|-------------------|
| (a) $(-10) - (-3) =$ | (h) $9 - (-3) =$ |
| (b) $4 - 6 =$ | (i) $7 - 5 =$ |
| (c) $16 - 12 =$ | (j) $7 - (-5) =$ |
| (d) $8 - (-2) =$ | (k) $2 - 9 =$ |
| (e) $(-8) - 2 =$ | (l) $2 - (-9) =$ |
| (f) $(-8) - (-2) =$ | (m) $3 - 10 =$ |
| (g) $(-9) - 2 =$ | (n) $3 - (-10) =$ |

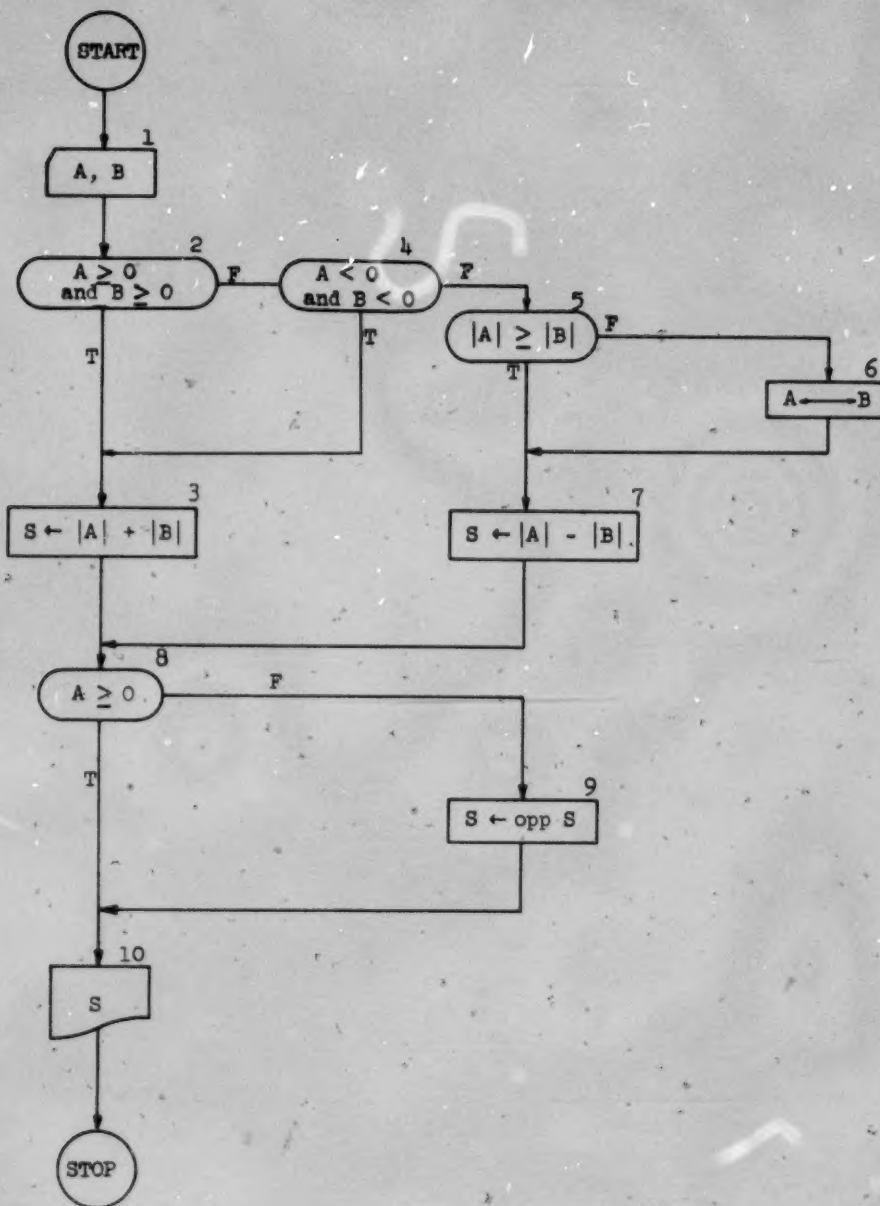
5. (a) From a temperature of 3° below zero, the temperature dropped 10° . What is the new temperature?
- (b) Write an equation to show how this question is related to subtraction of integers.
6. (a) The bottom of Death Valley is 282 feet below sea level. The top of Mt. Whitney, which is visible from Death Valley, is 14,495 feet above sea level. How high about the bottom of Death Valley is the top of Mt. Whitney?
- (b) Write an equation to show how this question is related to subtraction of integers.

5-8. A Flow Chart for Addition of Integers

Shown below is one possible flow chart for outputting the sum S of any two integers A , B . In order to construct this flow chart it is necessary to introduce assignment box 6. This assignment box,



with the uncommon assignment operator " $\longleftrightarrow$ ", commands us to interchange the values of the two variables.



Flow Chart for Addition of Integers: $S = A + B$.

Exercises 5-8

(Class Discussion)

1. Add the following pairs of integers by your usual method; then trace, mentally, each pair of integers through the flow chart to see if the sums are the same.

(a) $A \leftarrow 5, B \leftarrow -1$	(d) $A \leftarrow -5, B \leftarrow 1$
(b) $A \leftarrow -1, B \leftarrow 5$	(e) $A \leftarrow -5, B \leftarrow 0$
(c) $A \leftarrow 1, B \leftarrow -5$	(f) $A \leftarrow -5, B \leftarrow 5$
 2. Does the order in which we input values for the variables A, B, change the outputs?
 3. Suppose you are given any three integers and asked to find their sum S. Explain how this can be done by using the flow chart.
 4. Think about the algorithm you use to find the sum of any two integers. Is the flow chart a workable model for your algorithm?
-

5-9. Multiplication of Integers

We know that there are real life problems involving the product of two positive integers. These are problems that are familiar to us from arithmetic. We may ask, "Are there real life problems in which we would like to find products involving negative integers as factors?" The answer is, "There are." Here is an example showing such a situation.

Mr. Jones, an appliance dealer, wanted to clear out his stock of a certain line of appliances to prepare for new models. So he had a sale, reducing prices to a level at which he thought he could sell the appliances quickly. The prices were reduced so much that on some items he actually suffered a loss, while on others he still made a small profit. The following table shows the total number of each kind sold (column 2) and his profit or loss on each kind of item (column 3).

(1) Type of Appliance	(2) Number Sold	(3) Profit or Loss on Each Item (in dollars)	(4) Total Profit of Total Loss on this Type of Appliance
Refrigerators	4	22	88
Stoves	4	-22	-88
Dishwashers	5	-21	
Clothes Washers	3	-12	
Dryers	3	12	
Disposals	2	-11	
Freezers	4	25	

Notice that in column 3 a profit is indicated by a positive integer while a loss is indicated by a negative integer. To find the total profit on refrigerators, we can think of a repeated profit of 22, happening four times, once for each refrigerator sold. From our knowledge of addition of integers we know that

$$22 + 22 + 22 + 22 = 88.$$

Therefore, the total profit on refrigerators is represented by 88.

In the same manner, to find the total loss on stoves, we can think of a repeated loss of -22, happening four times, once for each stove sold. Again, from our knowledge of addition of integers we know that

$$(-22) + (-22) + (-22) + (-22) = -88.$$

Therefore, the total loss on stoves is represented by -88.

In finding the total profit on refrigerators and the total loss on stoves, we were able to express each problem in terms of repeated addition. But, our experience with arithmetic should remind us that it is possible to replace repeated addition with multiplication. Now, we see that either of the following sentences describes the total profit on refrigerators:

$$22 + 22 + 22 + 22 = 88;$$

$$4 \cdot 22 = 88.$$

From this, it seems reasonable to decide that

$$4 \cdot 22 = 88.$$

Thinking in the same way as above, we see that the total loss on stoves can now be described by either of the following sentences:

$$(-22) + (-22) + (-22) + (-22) = -88;$$

$$(-22) + (-22) + (-22) + (-22) = 4 \cdot (-22).$$

Again, from this, it seems reasonable to decide that

$$4 \cdot (-22) = -88.$$

Exercises 5-9a

(Class Discussion)

The following questions refer to entries in the table.

1. On which items did Mr. Jones show a profit?
2. Find the total profit for each item.
3. On which items did Mr. Jones show a loss?
4. Find the total loss for each item.
5. Did Mr. Jones make a profit or suffer a loss in his clearance sale?
How much?

When studying mathematics we often find that a model of a certain situation will help us to better understand the problem. The model shown below is the graph of the function

$$f : x \rightarrow 2x.$$

Careful observation of the graph and the table should tend to confirm what we have discovered so far about products involving negative integers as factors.

$f : x \rightarrow 2x$	
Input	Output
x	$2 \cdot x = 2x$
0	$2 \cdot 0 = 0$
4	$2 \cdot 4 = 8$
2	$2 \cdot 2 = 4$
(-1)	$2 \cdot (-1) =$
(-3)	$2 \cdot (-3) =$
(-5)	
(-8)	

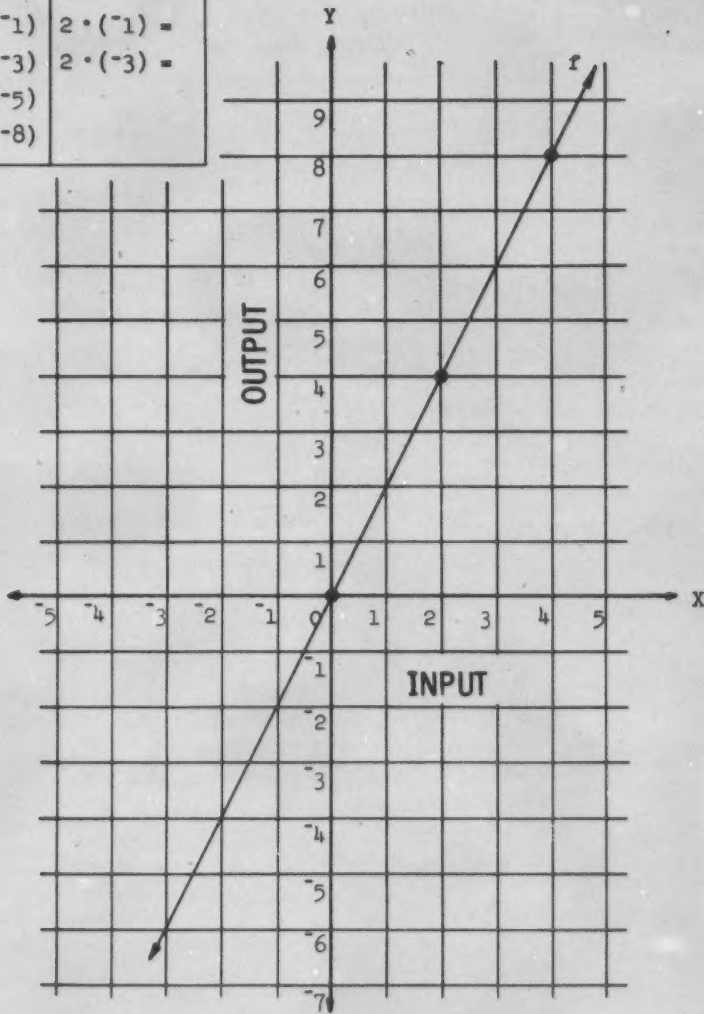


Figure 8

Exercises 5-9b

(Class Discussion)

1. When drawing the graph of the function $f : x \rightarrow 2x$ the points whose coordinates are $(0,0)$ and $(4,8)$ were plotted.
 - (a) Why wasn't it necessary to plot any more points in order to draw this line?
 - (b) Can you think of a good reason for letting zero be the first input value in the table?
2. After determining the line, the point whose coordinates are $(2,4)$ was plotted. Can you think of a good reason for plotting this additional point?
3. Notice that the line which is the graph of the function $f : x \rightarrow 2x$ passes through quadrant III. Now, let's read this graph in order to begin completing the table shown in Figure 8.
 - (a) If the input is -1 , then the output is ? .
 - (b) If the input is -3 , then the output is ? .
 - (c) If the input is -5 , then the output is ? .
4. If the input is -8 , then the output is ? .
5. From reading the graph in Figure 8, as well as from what you have previously learned, complete the following statement:

The product of a positive integer times a negative integer will be a ? integer.

Exercises 5-9c

1. Find the following sums:

(a) $(-1) + (-1) + (-1) + (-1) =$

(b) $(-3) + (-3) + (-3) + (-3) + (-3) =$

(c) $(-15) + (-15) + (-15) + (-15) =$

(d) $(-25) + (-25) + (-25) + (-25) + (-25) =$

(e) $(-7) + (-7) + (-7) + (-7) + (-7) + (-7) + (-7) =$

2. Each problem in Exercise 1 is an example of repeated addition. Replace each problem with multiplication, and find the product. Check your answers against those of Exercise 1.

Example: $(-8) + (-8) + (-8) = 3 \cdot (-8) = -24$.

3. Find the following products:

(a) $7 \cdot (-4)$

(e) $3 \cdot (-24)$

(i) $872 \cdot (-1)$

(b) $3 \cdot (-4)$

(f) $8 \cdot (-9)$

(j) $426 \cdot (0)$

(c) $5 \cdot (-16)$

(g) $15 \cdot (-4)$

(k) $12 \cdot (-12)$

(d) $21 \cdot (-35)$

(h) $6 \cdot (-8)$

(l) $100 \cdot (-5)$

4. Find the unknown factor in each of the following sentences.

(a) $3 \cdot \underline{\quad} = -36$

(d) $3 \cdot \underline{\quad} = -30$

(b) $5 \cdot \underline{\quad} = -75$

(e) $12 \cdot \underline{\quad} = -144$

(c) $2 \cdot \underline{\quad} = -10$

(f) $7 \cdot \underline{\quad} = -63$

When we studied addition of integers we found that the order in which we added any two integers did not change their sum (The Commutative Property of Addition). Since we know that we can replace repeated addition with multiplication, it seems reasonable to suppose that the order in which we multiply any two integers will not change their product (The Commutative Property of Multiplication). With this thought in mind, let us consider the graph of the function

$$g : x \rightarrow (-2)x.$$

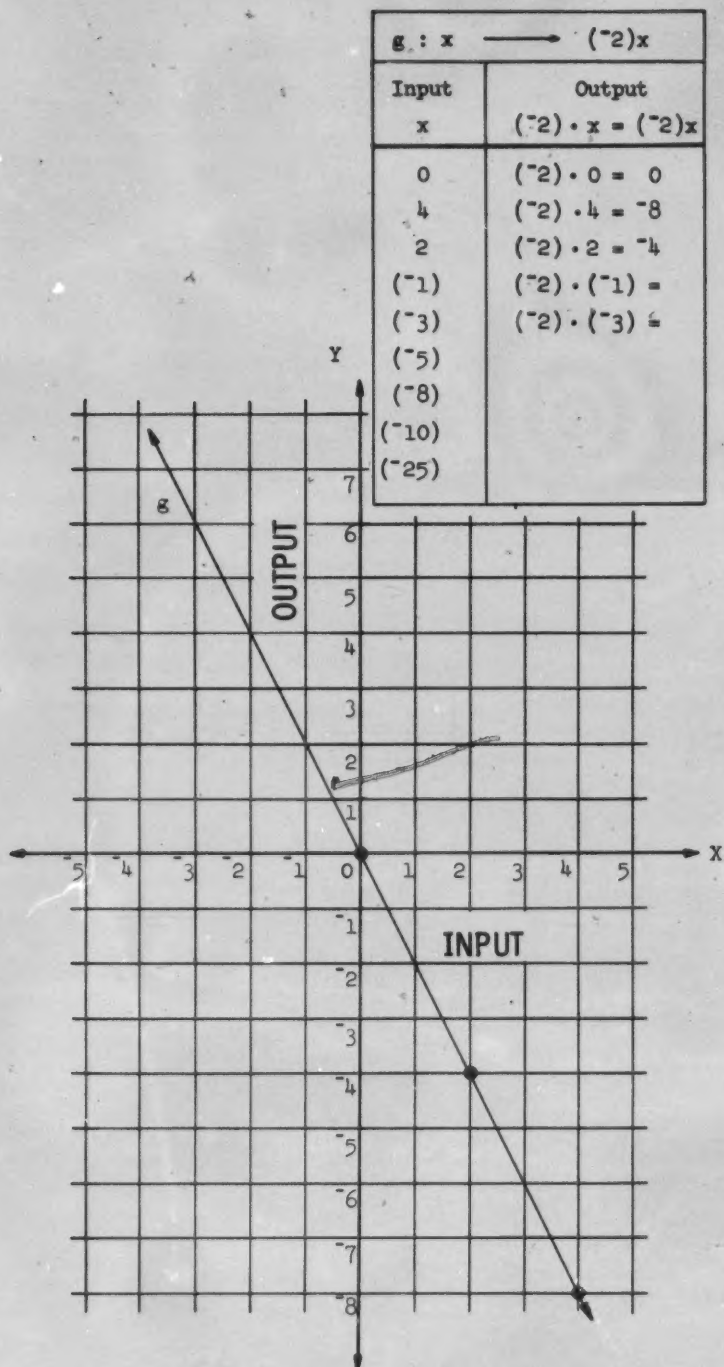


Figure 9. Multiplication by -2 represented graphically

Exercises 5-9d

(Class Discussion)

1. How many points was it necessary to plot in order to determine the graph of the function $g : x \rightarrow (-2)x$ shown in Figure 9.
2. After determining the line, the point whose coordinate is $(2, -4)$ was plotted. Why?
3. Now, let's read the graph shown in Figure 9.
 - (a) If the input is 1, then the output is ?.
 - (b) If the input is 3, then the output is ?.
 - (c) If the input is 5, then the output is ?.
4. Now, let's read the graph again in order to begin completing the table shown in Figure 9.
 - (a) If the input is -1, then the output is ?.
 - (b) If the input is -3, then the output is ?.
 - (c) If the input is -5, then the output is ?.
5.
 - (a) If the input is -8, then the output is ?.
 - (b) If the input is -10, then the output is ?.
 - (c) If the input is -25, then the output is ?.
6. From your observation of the graph of the function

$$g : x \rightarrow (-2)x$$

it appears that if x is positive ($x > 0$), then $(-2) \cdot x < 0$,
and that if x is negative ($x < 0$), then $(-2) \cdot x > 0$.

Suppose we have a function such that $h : x \rightarrow ax$, where a is
any negative integer ($a < 0$). Do you think it will always be true
that

$$a \cdot x < 0 \text{ if } x \text{ is positive } (x > 0),$$

and that

$$a \cdot x > 0 \text{ if } x \text{ is negative } (x < 0)?$$

7. Complete the following statements.

- (a) The product of a negative integer times a positive integer will be a _____ integer.
- (b) The product of a negative integer times a negative integer will be a _____ integer.
- _____

Exercises 5-9e

1. Find the product for each of the problems (a) through (l).

- | | | |
|---------------------|---------------------|-----------------------|
| (a) $6 \cdot 9$ | (e) $0 \cdot 192$ | (i) $(-1) \cdot 59$ |
| (b) $75 \cdot 1$ | (f) $6 \cdot (-9)$ | (j) $(-1) \cdot (-1)$ |
| (c) $1 \cdot (-37)$ | (g) $18 \cdot (-1)$ | (k) $(-6) \cdot (-5)$ |
| (d) $(-6) \cdot 0$ | (h) $(-6) \cdot 4$ | (l) $(-9) \cdot (-8)$ |
- (m) Do you think that the product of any two integers will always be an integer? If not, find a counterexample.

2. Find the product for each of the problems (a) through (l).

- | | | |
|--------------------|-----------------------|-----------------------|
| (a) $7 \cdot 3$ | (e) $(-1) \cdot (29)$ | (i) $(-4) \cdot (-3)$ |
| (b) $3 \cdot 7$ | (f) $29 \cdot (-1)$ | (j) $(-3) \cdot (-4)$ |
| (c) $8 \cdot (-7)$ | (g) $0 \cdot (-16)$ | (k) $(-9) \cdot (-2)$ |
| (d) $(-7) \cdot 8$ | (h) $(-16) \cdot 0$ | (l) $(-2) \cdot (-9)$ |
- (m) Do you think that the order in which we multiply any two integers will change the product? If your answer is yes, write a counterexample.

3. Find the product for each of the problems (a) through (h).
Perform the multiplication within the parentheses first.

- | | |
|-------------------------------------|----------------------------------|
| (a) $(37 \cdot 25) \cdot 4$ | (e) $(12 \cdot (-2)) \cdot 6$ |
| (b) $37 \cdot (25 \cdot 4)$ | (f) $12 \cdot ((-2) \cdot 6)$ |
| (c) $((-2) \cdot (-5)) \cdot (-17)$ | (g) $((-5) \cdot (-20)) \cdot 7$ |
| (d) $(-2) \cdot ((-5) \cdot (-17))$ | (h) $(-5) \cdot ((-20) \cdot 7)$ |
- (i) Do you think that the manner in which we group the middle factor, when we multiply any three integers, will change the product? If

your answer is yes, write a counterexample.

4. Do you think that the product of one and any integer, a, is a?
If not, find a counterexample.
5. Find the unknown factor in each of the following sentences.
- | | |
|-----------------------------|-----------------------------|
| (a) $(?) \cdot 6 = -12$ | (i) $1 \cdot (?) = -1$ |
| (b) $5 \cdot (?) = -15$ | (j) $6 \cdot (?) = -36$ |
| (c) $(-10) \cdot (?) = 100$ | (k) $(-9) \cdot (?) = 81$ |
| (d) $(-5) \cdot (?) = 20$ | (l) $5 \cdot (?) = -30$ |
| (e) $(-5) \cdot (?) = -20$ | (m) $(?) \cdot (-10) = -90$ |
| (f) $11 \cdot (?) = -110$ | (n) $(?) \cdot (-50) = 100$ |
| (g) $(-1) \cdot (?) = 1$ | (o) $(-6) \cdot (?) = -60$ |
| (h) $(-7) \cdot (?) = 0$ | (p) $(-1) \cdot (?) = -1$ |
6. If two integers, a and b, are multiplied what do we know about them if their product is:
- (a) positive?
 - (b) zero?
 - (c) negative?

By now, your work with multiplication of integers should have led you to the following conclusions:

- (a) If two integers are both positive or both negative, then their product is positive.
- (b) If one integer is positive and another integer is negative, then their product is negative.

5-10. A New Approach to Multiplication of Integers

In your earlier work with multiplication you found that the integer 1 behaved in a special manner. That is, the product of 1 and any integer is that integer.

For example: $1 \cdot 827 = 827,$

$$1 \cdot 5 = 5,$$

$$1 \cdot 0 = 0,$$

$$1 \cdot (-17) = -17.$$

Actually, we can see that 1 used as one of two factors "identifies" the other factor as the product. Because of this behavior we often call 1 "the identity element for multiplication".

Earlier, when talking about opposites, we found that

$$\text{opp } 1 = -1 \text{ and } \text{opp } (-1) = 1.$$

Since the integer 1 behaves in a special way when it is used as a factor, we should not be surprised to learn that the integer opp 1 behaves in an opposite way when it is used as a factor. That is, the product of opp 1 and any integer is the opposite of that integer!

For example: $1 \cdot 5 = 5,$ and

$$\text{opp } 1 \cdot 5 = \text{opp } 5 = -5.$$

$$1 \cdot (-7) = -7, \text{ and}$$

$$\text{opp } 1 \cdot (-7) = \text{opp } (-7) = 7.$$

In general:

$$1 \cdot a = a, \text{ and}$$

$$\text{opp } 1 \cdot a = \text{opp } a.$$

Exercises 5-10a

(Class Discussion)

1. Now, we are going to multiply a positive integer by a negative integer. In the left-hand column, below, are statements showing each step of the process. In the right-hand column, you are to provide, in your own words, a reason for each step. Step 1 is done for you.

Statements	Reasons
1. $(-3) \cdot 5 = (\text{opp } 3) \cdot 5$	1. "Negative 3" and "the opposite of 3" name the same number.
2. $= (\text{opp } 1 \cdot 3) \cdot 5$	2. Why?
3. $= \text{opp } 1 \cdot (3 \cdot 5)$	3. Why?
4. $= \text{opp } 1 \cdot 15$	4. Why?
5. $= \text{opp } 15$	5. Why?
6. $= -15$	6. Why?

2. At this time, we are going to find the product of two negative integers. Again, the left-hand column will show each step of the process. You are to provide, in your own words, a reason for each step.

Statements	Reasons
1. $(-3) \cdot (-5) = \text{opp } 3 \cdot (-5)$	1. "-3" and "opp 3" name the same number.
2. $= \text{opp } 3 \cdot \text{opp } 5$	2. Why?
3. $= (\text{opp } 1 \cdot 3) \cdot \text{opp } 5$	3. Why?
4. $= \text{opp } 1 \cdot (3 \cdot \text{opp } 5)$	4. Why?
5. $= \text{opp } 1 \cdot (\text{opp } 5 \cdot 3)$	5. Why?
6. $= \text{opp } 1 \cdot ((\text{opp } 1 \cdot 5) \cdot 3)$	6. Why?
7. $= \text{opp } 1 \cdot (\text{opp } 1 \cdot (5 \cdot 3))$	7. Why?
8. $= \text{opp } 1 \cdot (\text{opp } 1 \cdot 15)$	8. Why?
9. $= \text{opp } 1 \cdot \text{opp } 15$	9. Why?
10. $= \text{opp } 1 \cdot (-15)$	10. Why?
11. $= \text{opp } (-15)$	11. Why?
12. $= 15$	12. Why?

3. Given two integers, a and b, answer the following questions concerning their product.

(a) If a is a positive integer ($a > 0$) and b is a positive

- integer ($b > 0$), then the product $(a \cdot b)$ will be a ?
integer.
- (b) If a is a positive integer ($a > 0$) and b is a negative integer ($b < 0$), then the product $(a \cdot b)$ will be a ?
integer.
- (c) If a is a negative integer ($a < 0$) and b is a positive integer ($b > 0$), then the product $(a \cdot b)$ will be a ?
integer.
- (d) If a is a negative integer ($a < 0$) and b is a negative integer ($b < 0$), then the product $(a \cdot b)$ will be a ?
integer.

Exercises 5-10b

1. Find the products:

- | | |
|-------------------------|----------------------------------|
| (a) $(-6) \cdot (-10)$ | (n) $(-16) \cdot (-12)$ |
| (b) $(-3) \cdot (-4)$ | (o) $(-45) \cdot (-3)$ |
| (c) $5 \cdot 6$ | (p) $25 \cdot (-3)$ |
| (d) $(-3) \cdot (-6)$ | (q) $(-27) \cdot 0$ |
| (e) $(-8) \cdot (-4)$ | (r) $(-16) \cdot 1$ |
| (f) $(-75) \cdot (-4)$ | (s) $20 \cdot (-10) \cdot (-5)$ |
| (g) $(-4) \cdot (-10)$ | (t) $(-3) \cdot (-5) \cdot (-4)$ |
| (h) $4 \cdot (-10)$ | (u) $(-5) \cdot 6 \cdot (-2)$ |
| (i) $(-10) \cdot 4$ | (v) $(-4) \cdot (-5) \cdot 3$ |
| (j) $(-6) \cdot (-7)$ | (w) $(-2) \cdot (-1) \cdot (-3)$ |
| (k) $(-15) \cdot (-4)$ | (x) $(-4) \cdot (-2) \cdot 2$ |
| (l) $(-20) \cdot (-10)$ | (y) $(-3) \cdot (-3) \cdot (-3)$ |
| (m) $16 \cdot (-12)$ | (z) $(-2) \cdot 2 \cdot (-2)$ |

2. Complete the table:

x	$(-4) \cdot x$
3	-8
1	0
-1	
-2	12
-4	100

3. Find the unknown factor in each of the following sentences.

(a) $3 \cdot \underline{\quad} = 15$

(f) $(-8) \cdot \underline{\quad} = 8$

(b) $5 \cdot \underline{\quad} = -20$

(g) $(-9) \cdot \underline{\quad} = -81$

(c) $(-2) \cdot \underline{\quad} = 6$

(h) $(-8) \cdot \underline{\quad} = 64$

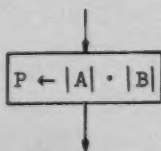
(d) $(-3) \cdot \underline{\quad} = -3$

(i) $(-7) \cdot \underline{\quad} = 63$

(e) $(-7) \cdot \underline{\quad} = 0$

5-11. A Flow Chart for Multiplication of Integers

It is very easy to construct a flow chart for outputting the product P of any two integers, A , B . If the variables A and B represent the two factors, then the computation of the absolute value of the product P is given by:



The test to determine whether the two factors are both non-negative or both negative makes use of two decision boxes, as pictured below.



If, in either case the statement is true, we merely output the above computed value of P as our answer and stop.



If, in both cases the statement is false, then we replace P by its opposite

$$P \leftarrow \text{opp } P$$

and output the new value of P .

The entire flow chart is:

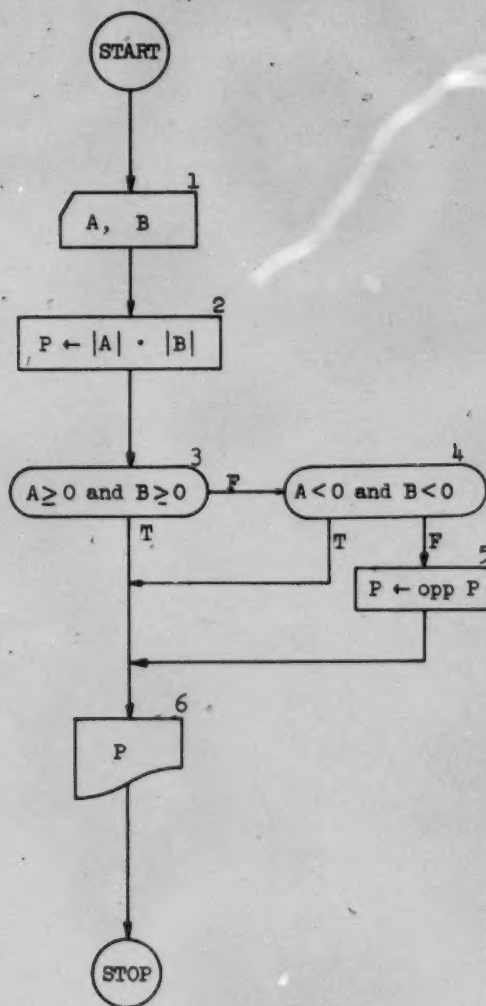


Figure 10

Exercises 5-11

(Class Discussion)

1. Multiply the following pairs of integers by your usual method; then trace, mentally, each pair of integers through the flow chart to see if the products are the same.

(a) $4 \cdot (-5)$	(e) 0.285
(b) $(-5) \cdot (4)$	(f) $(-7) \cdot (-9)$
(c) $1 \cdot (-17)$	(g) $(-9) \cdot (-7)$
(d) $(-1) \cdot 17$	(h) $8 \cdot 8$
 2. Does the order in which we input values for the variables A, B, change the outputs?
 3. Suppose you are given any three integers and asked to find their product P. Explain how this can be done by using the flow chart.
 4. Think about the algorithm you use to find the product of any two integers. Is the flow chart a workable model for your algorithm?
-

5-12. Organizing Our Knowledge

Much of the work we have been doing so far has consisted of performing the operations of addition and multiplication on the set of integers. In learning to perform these operations we also learned a great deal about the behavior of integers. Because we have introduced you to so many different, though related, ideas in this chapter, let's stop for a moment to see whether these many ideas can be organized or structured.

Remember, we wish to organize the characteristics or properties that the integers seem to have when they are either added or multiplied. In doing this, we will name these properties, but you should realize that these names are always secondary in importance to the behavior which they describe. The names are useful, though, for identifying the particular behavior we may wish to talk about.

The Closure Property tells us that if we add or multiply any two integers the result will always be an integer.

The Commutative Property tells us that the order in which we add, or multiply any two integers will not change the result.

The Associative Property tells us that the manner in which we group the middle integer, when we add, or multiply, any three integers, will not change the result.

The Properties of Zero and One tell us that the sum of any integer and 0, or the product of 1 and any integer, is that integer. That is, the integer remains unchanged or keeps its identity!

The Property of Opposites tells us that the sum of an integer and its opposite is 0, the identity element for addition.

Perhaps the best way to emphasize these properties is by means of a table. Notice that, with only one exception, for each property of addition there is a corresponding property of multiplication.

a, b, and c represent integers.

Closure Property of Addition

$a + b$ is an integer

Example

$3 + (-5) = -2$ and -2 is an integer

Closure Property of Multiplication

$a \cdot b$ is an integer

Example

$3 \cdot (-5) = -15$ and -15 is an integer

Commutative Property of Addition

$a + b = b + a$

Example

$3 + (-5) = (-5) + 3$

Commutative Property of Multiplication

$a \cdot b = b \cdot a$

Example

$3 \cdot (-5) = (-5) \cdot 3$

Associative Property of Addition

$(a + b) + c = a + (b + c)$

Example

$(3 + (-5)) + 5 = 3 + ((-5) + 5)$

Associative Property of Multiplication

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Example

$(3 \cdot (-5)) \cdot 5 = 3 \cdot ((-5) \cdot 5)$

Addition Property of Zero

$a + 0 = a$

Example

$(-3) + 0 = -3$

Multiplication Property of One

$1 \cdot a = a$

Example

$1 \cdot (-3) = -3$

Addition Property of Opposites

$a + \text{opp } a = 0$

Example

$3 + (-3) = 0$

?

Exercises 5-12

(Class Discussion)

1. Is the set of even integers closed under:
 - (a) addition?
 - (b) multiplication?
2. Is the set of odd integers closed under:
 - (a) addition?
 - (b) multiplication?
3. Which of the defined operations below are commutative?
 - (a) "D" means to find the sum of the first and twice the second.
Example: $3 \text{ D } 5 = 3 + (2 \cdot 5) = 13$.
 - (b) "Z" means to find the sum of the first and the product of the first and the second. Example: $4 \text{ Z } 7 = 4 + (4 \cdot 7) = 32$.
 - (c) "F" means to find the product of the first and one more than the second. Example: $8 \text{ F } 0 = 8 \cdot 1 = 8$.
 - (d) "Q" means to find three times the sum of the first and the second. Example: $8 \text{ Q } 5 = 3 \cdot (8 + 5) = 39$.
4. Rewrite these problems using the associative and commutative properties whenever they make the operation easier. Then, find the answers.
Example: $25 + (36 + 75) =$
 $25 + (75 + 36) = \text{Commutative Property of Addition}$
 $(25 + 75) + 36 = \text{Associative Property of Addition}$
 $100 + 36 =$
 136
 - (a) $(6 + (-1)) + (-9)$
 - (b) $2 \cdot (13 \cdot 10)$
 - (c) $(12 \cdot 9) \cdot 10$
 - (d) $(-4) \cdot ((-25) \cdot 76)$

(e) $(-340) + (522 + (-60))$

(f) $(5 \cdot 67) \cdot 2$

5. Identify those sentences below which are always true when the variable is replaced by any integer.

(a) $w + 0 = w$

(b) $0 + w = w$

(c) $w - 0 = w$

(d) $0 \cdot w = 0$

(e) $w \cdot 0 = 0$

(f) If $u \cdot w = 0$, then either u or w is zero, or both are zero.

6. Give a good reason why one is called "the identity element for multiplication".
-

5-13. The Link Between Addition and Multiplication

We have two kinds of evidence to suggest that the operations of addition and multiplication are closely related to each other:

- (1) we can replace repeated addition with multiplication, and
- (2) with only one exception, for each property of addition there is a corresponding property of multiplication.

Now, we are going to introduce you to the property which directly connects these two operations. This property is remarkably useful and is illustrated by the following sentence:

$$4 \cdot (5 + 7) = (4 \cdot 5) + (4 \cdot 7).$$

The sentence states that " $4 \cdot (5 + 7)$ " and " $(4 \cdot 5) + (4 \cdot 7)$ " are names for the same number. We will show you that this is true.

$$\begin{aligned} 4 \cdot (5 + 7) &= 4 \cdot 12 = 48, \text{ and} \\ (4 \cdot 5) + (4 \cdot 7) &= 20 + 28 = 48. \end{aligned}$$

Notice that when reading the sentence " $4 \cdot (5 + 7) = (4 \cdot 5) + (4 \cdot 7)$ " from left to right, it seems that the 4, in $4 \cdot (5 + 7)$, has been

distributed to both of the numbers within the parentheses (5 + 7). For this reason, the property is named

The Distributive Property of Multiplication over Addition.

The Distributive Property tells us that if a, b, and c represent integers, then

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

The distributive property can also be written with the expression " $(a \cdot b) + (a \cdot c)$ " on the left side and the expression " $a \cdot (b + c)$ " on the right side. That is, we can also state the property as follows:

$$(a \cdot b) + (a \cdot c) = a \cdot (b + c).$$

In trying to understand the distributive property it is very useful to work with an array. Let's consider an a by (b + c) array, as is pictured below.

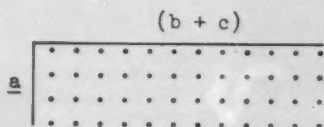


Figure 11

Here, a refers to the number of rows (in this case, 4), and (b + c) refers to the number of members in each row (in this case, 12). It is clear that this array may be separated, as is shown below,

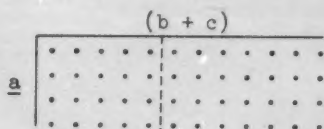


Figure 12

and, then, displayed in the following way.

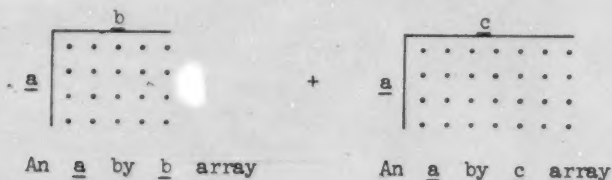


Figure 13

As a result, the number of members in the array of Figure 11, $a \cdot (b + c)$, is the sum of the number of members in the arrays of Figure 13, $(a \cdot b) + (a \cdot c)$. That is,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

We can verify this conclusion by noticing that in Figure 11 a represents the integer 4 and (b + c) represents the integer 12. Hence, Figure 11 shows a $(4 \cdot 12)$ array. In Figure 12 we have renamed 12 as $(5 + 7)$, and in Figure 13 we see two arrays, a $(4 \cdot 5)$ array and a $(4 \cdot 7)$ array. Thus, we can reason that a $(4 \cdot 12)$ array has the same number of members as a $(4 \cdot 5)$ array plus a $(4 \cdot 7)$ array. That is,

$$4 \cdot 12 = 4 \cdot (5 + 7) = (4 \cdot 5) + (4 \cdot 7).$$

Exercises 5-13a

(Class Discussion)

- Each of the following expressions names an integer. Use the distributive property in order to find the common name for each integer.

- $((-9) \cdot 17) + ((-9) \cdot 83)$
- $((-25) \cdot 17) + (26 \cdot 17)$
- $(854 \cdot 673) + (146 \cdot 673)$
- $(84 \cdot (-367)) + (84 \cdot (-633))$
- $(55,428 \cdot (-3468)) + (3468 \cdot 55,428)$

Exercises 5-13b

- In each of the following, verify that the two numerals name the same number.

- | | |
|------------------------------------|------------------------------------|
| (a) $(3 \cdot 2) \cdot (-4)$ | and $3 \cdot (2 \cdot (-4))$ |
| (b) $(3 \cdot (-2)) \cdot (-4)$ | and $3 \cdot ((-2) \cdot (-4))$ |
| (c) $(3 \cdot (-2)) \cdot 4$ | and $3 \cdot ((-2) \cdot 4)$ |
| (d) $((-3) \cdot 2) \cdot (-4)$ | and $(-3) \cdot (2 \cdot (-4))$ |
| (e) $((-3) \cdot (-2)) \cdot (-4)$ | and $(-3) \cdot ((-2) \cdot (-4))$ |
| (f) $(3 \cdot (-2)) \cdot 0$ | and $3 \cdot ((-2) \cdot 0)$ |

2. Find common names for each of the following.
- (a) $(-5) \cdot 17 \cdot (-20) \cdot 4$ (c) $(-1) \cdot (-9) \cdot 1 \cdot 9$
 (b) $(-7) \cdot (-25) \cdot 3 \cdot (-4)$ (d) $(-325) \cdot (-257) \cdot (-27) \cdot 0$
3. Use the distributive property, if convenient, in order to find common names for the following.
- (a) $(-9) \cdot (-92) + (-9) \cdot (-8)$ (c) $57 \cdot (136 + (-36))$
 (b) $(63 \cdot 6) + (137 \cdot 6)$ (d) $(12 + (-11)) \cdot 78$
4. Use the distributive property to rename the following as indicated sums. Example: $6 \cdot (10 + 5) = (6 \cdot 10) + (6 \cdot 5)$.
- (a) $3 \cdot (x + 5)$ (d) $(-3) \cdot (x + (-3))$
 (b) $a \cdot (7 + (-5))$ (e) $(-7) \cdot ((-5) + (-10))$
 (c) $(-9) \cdot (a + 7)$ (f) $25 \cdot ((-4) + (-6))$
5. Use the distributive property to rename the following as indicated products. Example: $(3 \cdot 5) + (3 \cdot 7) = 3 \cdot (5 + 7)$.
- (a) $(5 \cdot 7) + (5 \cdot 9)$ (d) $(2 \cdot 7) + ((-2) \cdot 13)$
 (b) $(6 \cdot 8) + (6 \cdot a)$ (e) $(3 \cdot x) + (4 \cdot x)$
 (c) $(a \cdot 7) + (a \cdot 13)$ (f) $(347 \cdot 8) + (2 \cdot 347)$
-

5-14. Summary

Section 5-1.

The number line helps us to organize and to expand our knowledge of number.

Section 5-2.

In the construction of the number line, a correspondence is established from a set of numbers to a set of points, and this correspondence is a function.

$\{1, 2, 3, 4, 5, \dots\}$ is called the set of positive integers.

$\{0, 1, 2, 3, \dots\}$ is called the set of non-negative integers.

$\{\dots, -3, -2, -1\}$ is called the set of negative integers.

$\{\dots, -2, -1, 0, 1, \dots\}$ is called the set of integers.

If one integer is greater than ($>$) another, then it is to the right of the other on the horizontal number line.

If one integer is less than ($<$) another, then it is to the left of the other on the horizontal number line.

Section 5-3.

Corresponding to each integer there is exactly one integer, its opposite.

The opposite function can be described by $f : x \rightarrow \text{opp } x$.

Section 5-4.

Corresponding to each integer there is exactly one undirected distance from zero, and this distance is called the absolute value of the integer.

The absolute value function can be described by $f : x \rightarrow |x|$.

Section 5-5.

A non-zero integer can be represented on the number line by an arrow:

- (1) the length of the arrow corresponds to the absolute value of the integer;
- (2) the arrow will point right if the integer is positive, left if the integer is negative.

Section 5-6.

With respect to the set of integers the operation of addition has important properties, named as follows:

- (1) The Closure Property of Addition,
- (2) The Commutative Property of Addition,
- (3) The Associative Property of Addition,
- (4) The Addition Property of Zero,
- (5) The Addition Property of Opposites.

Section 5-7.

The subtraction of one integer from another can always be expressed in terms of addition! That is, if a and b represent any two integers, then $a - b = a + \text{opp } b$.

Section 5-8.

It is possible to construct a flow chart which will output the sum S of any two integers A, B.

Section 5-9.

If two integers are both positive or both negative, then their product is positive.

If one integer is positive and another integer is negative, then their product is negative.

With respect to the set of integers the operation of multiplication has important properties, named as follows:

- (1) The Closure Property of Multiplication,
- (2) The Commutative Property of Multiplication,
- (3) The Associative Property of Multiplication,
- (4) The Multiplication Property of One.

Section 5-10.

The integers 1 and opp 1 behave in opposite ways when they are used as factors. That is, if a represents an integer, then

$$\begin{aligned} 1 \cdot a &= a, \text{ and} \\ \text{opp } 1 \cdot a &= \text{opp } a. \end{aligned}$$

Section 5-11.

It is possible to construct a flow chart which will output the product P of any two integers A, B.

Section 5-12.

With respect to the set of integers, the operations of addition and multiplication have important properties. See the table on page 39.

Section 5-13.

The distributive property joins the operations of addition and multiplication. For example,

$$(-2) \cdot (5 + (-4)) = (-2) \cdot 5 + (-2) \cdot (-4).$$

That is, $(-2) \cdot (5 + (-4)) = (-2) \cdot 1 = -2$, and

$$(-2) \cdot 5 + (-2) \cdot (-4) = (-10) + 8 = -2.$$

Chapter 6

RATIONAL NUMBERS

6-1. Introduction

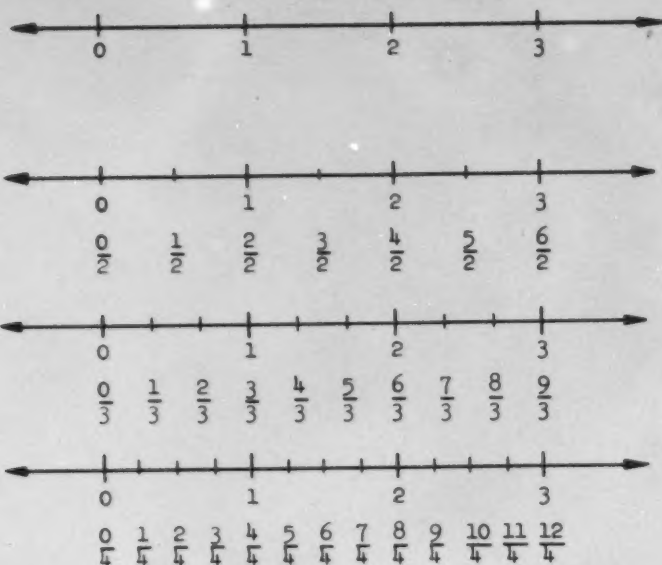
Our work with numbers so far has been done with the set of integers. We have pretended that they are the only numbers that exist, and we have examined their behavior under addition and multiplication.

In our previous development the number line was marked only at the points which correspond to integers, leaving gaps containing many points that do not correspond to numbers. Also, it may have occurred to you that we have not yet discussed "division". Using only the integers it should be clear that many "division" problems do not have an integer for an answer (for example $3 \div 4$). That is, the set of integers is not closed under the operation of division..

We will now extend our number system to try to fill in these gaps and to provide quotients for any meaningful division problems. In our extension of the number system to include what we will call rational numbers we will proceed much as we did with the integers. That is, we will start with the number line and develop some ideas, relations, and properties of rational numbers.

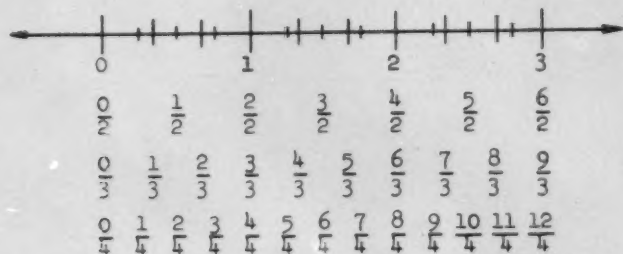
6-2. Extending the Number System

Just as we did when developing the integers, we will start with a line on which points are labeled with the non-negative integers. We can label other points by dividing the intervals into halves, thirds, fourths, etc., to obtain



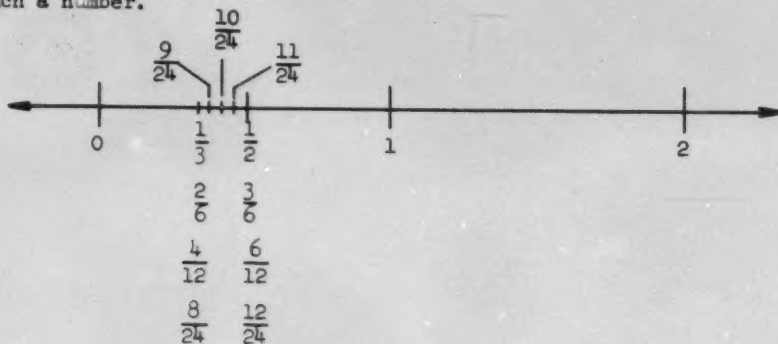
where we can think of an unending set of lines with the intervals divided into successively greater numbers of parts.

Now put all of these together on one line. We have a labeling of points corresponding to some of the members of a set of numbers which are called the non-negative rational numbers.



You may think we will use up the supply of points in this process of assigning numbers to points. Are we sure that between any two distinct points, no matter how close, there is another point? We can answer this question for points corresponding to rational numbers as follows: Choose two such points, for example, the points with coordinates $\frac{1}{3}$ and $\frac{1}{2}$. We know that $\frac{1}{3}$ has the names: $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, $\frac{8}{24}$, etc., and $\frac{1}{2}$ has the

names: $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, $\frac{12}{24}$, etc. Then a number between $\frac{1}{3}$ and $\frac{1}{2}$ is a number between $\frac{8}{24}$ and $\frac{12}{24}$. We can choose any one of $\frac{9}{24}$, $\frac{10}{24}$, $\frac{11}{24}$ as such a number.

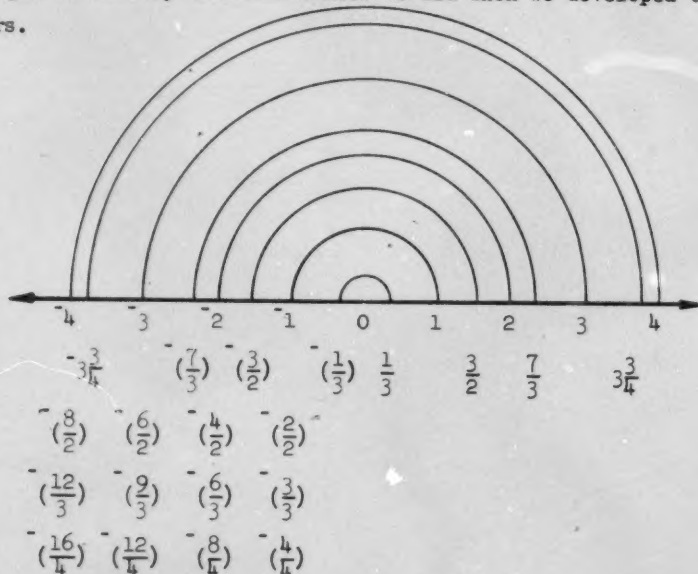


etc.

Thus, the points with coordinates $\frac{9}{24}$, $\frac{10}{24}$, $\frac{11}{24}$ are between the points with coordinates $\frac{1}{3}$ and $\frac{1}{2}$.

This process of finding a point between points by finding a number between numbers could be carried out for any two points, no matter how close. This serves to show us that there are not only a great many points between any two given points, but infinitely many points.

Now we can proceed to locate points on the half-line to the left of the origin in exactly the same manner we did when we developed the set of integers.



We now have a labeling of points corresponding to some of the members of a set of numbers which are called rational numbers.

From the above figure it is apparent that, like the integers, every rational number has an opposite. We would expect that they would also behave like the integers in that the sum of a rational number and its opposite results in the identity element for addition, 0. For example,

$$\frac{1}{2} + \text{opp } \frac{1}{2} = 0$$

$$-\frac{3}{4} + \text{opp } -\frac{3}{4} = 0.$$

You are familiar with names for numbers that are not integers and lie to the right of zero. You probably have called these numbers "fractions".

At this time let us review what is meant by a "fraction". Notice that the coordinate of the point on the number line corresponding to (-2) has many different names:

$$-(\frac{4}{2}), (-\frac{6}{3}), (-\frac{8}{4}), \text{ etc.}$$

Each of these symbols is a fraction and each is a different name for the same number (-2). The number named by " $2\frac{1}{2}$ " can also be represented by fractions: $\frac{5}{2}$, $\frac{10}{4}$, $\frac{15}{6}$, etc. Also, the number named by ".6" can be represented by: $\frac{3}{5}$, $\frac{6}{10}$, $\frac{9}{15}$, etc. In general, we shall mean by a fraction a symbol which indicates the quotient of two integers.

A number which can be (not must be) represented by a fraction indicating the quotient of two integers, excluding division by zero, is called a rational number. Thus, -2, $2\frac{1}{2}$, .6, $-\frac{3}{2}$, $\frac{1001}{37}$ are examples of rational numbers.

Check Your Reading

1. Between any two given points, no matter how close, how many other points are there?
2. What do we mean by a "fraction"?
3. What is a number called which can be represented as the quotient of two integers, excluding division by zero?

Exercises 6-2a

(Class Discussion)

If set $R = \{-5, -4, (-\frac{10}{3}), (-\frac{3}{2}), -1, 0, \frac{1}{4}, 1, \frac{3}{2}, 2, \frac{7}{3}, 6\}$,

1. List I, the set of integers in R.
 2. List W, the set of non-negative integers in R.
 3. List A, the set of positive integers in R.
 4. List P, the set of negative integers in R.
 5. List N, the set of positive rational numbers in R.
 6. List G, the set of negative rational numbers in R.
 7. Are all the members of set R rational numbers, that is, can each member of R be expressed as the quotient of two integers, excluding division by zero?
 8. Find four rational numbers between $\frac{3}{7}$ and $\frac{4}{7}$.
-

Exercises 6-2b

1. Place the symbols $>$, $<$, or $=$, between the following pairs of rational numbers in order to make the statement true.

(a) 5, 4	(e) $-(\frac{5}{2}), -(\frac{10}{4})$
(b) 0, -5	(f) $(-4), -(\frac{15}{4})$
(c) -4, -7	(g) $-(\frac{16}{3}), -(\frac{21}{4})$
(d) $-(\frac{1}{2}), 1$	(h) $-(2\frac{1}{2}), -(\frac{5}{2})$
2. Is -2
 - (a) an integer?
 - (b) a rational number?
3. Is $-\frac{10}{3}$
 - (a) an integer?
 - (b) a rational number?

4. Is $-\frac{12}{3}$
- (a) an integer?
 - (b) a rational number?
5. Is 0
- (a) an integer?
 - (b) a rational number?
 - (c) positive or negative?
6. Find three rational numbers between each of the following pairs of rational numbers.
- (a) $\frac{2}{4}, \frac{3}{4}$
 - (b) $\frac{1}{2}, \frac{3}{6}$
 - (c) $\frac{2}{3}, \frac{7}{9}$
 - (d) $\frac{1}{2}, \frac{1}{3}$
7. What is the opposite of:
- (a) $\frac{1}{2}$?
 - (b) $-\frac{3}{4}$?
 - (c) $3\frac{1}{4}$?
 - (d) $-4\frac{1}{2}$?
 - (e) opp $1\frac{1}{3}$?
 - (f) opp opp $15\frac{5}{7}$?
8. Which of the following sets are the same?
- (a) the set of whole numbers.
 - (b) the set of positive integers.
 - (c) the set of non-negative integers.
 - (d) the set of integers.
 - (e) the set of counting numbers.
-

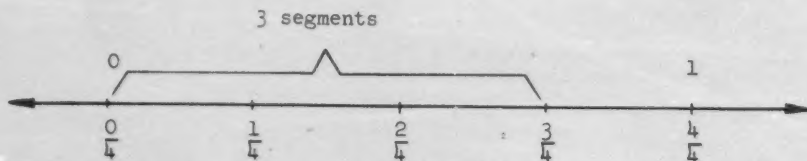
6-3. Another Interpretation of " $\frac{a}{b}$ "

In the last section we defined a rational number as one that can be (not must be) represented by a "fraction" indicating the quotient of two integers, excluding division by zero. Using symbols we can write that any number of the form

$$\frac{a}{b} \text{ (read } a \text{ divided by } b), b \neq 0, a \text{ and } b \text{ integers,}$$

is a rational number.

There is a second way in which we can think of the symbol " $\frac{a}{b}$ ". Consider the number line model below for the rational number " $\frac{3}{4}$ ". Divide the unit segment into four segments of equal length. Now count off three of these segments.



We have now located the coordinate of the point corresponding to the rational number " $\frac{3}{4}$ ". In the process of counting we are actually finding the sum

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4}, \text{ and our knowledge of repeated addition tells}$$

us that

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} \text{ can be expressed as } 3 \cdot \frac{1}{4}, \text{ therefore}$$

$$\frac{3}{4} = 3 \cdot \frac{1}{4}.$$

Using this same idea, we can see that

$$\frac{4}{5} = 4 \cdot \frac{1}{5}; \quad \frac{8}{5} = 8 \cdot \frac{1}{5}; \quad \frac{25}{25} = 25 \cdot \frac{1}{25}; \text{ and so on.}$$

We should be able to conclude then, that the symbol " $\frac{a}{b}$ " beside being an indicated division can also be thought of as an indicated product where b, called the denominator, tells you how many segments of equal length the unit has been divided into; while a, the numerator, tells you how many of

these segments we are considering, that is, if $b \neq 0$, then

$$\frac{a}{b} = a \cdot \frac{1}{b}.$$

Exercises 6-3

(Class Discussion)

1. A rational number can be represented by a fraction. Must it be represented by a fraction?
2. The set of all integers is a subset of the set of all rational numbers; that is, all integers are rational numbers. Are all rational numbers integers?
3. There are many possible names for the same number. What are some other names for 1? $(-\frac{1}{2})$? (-8) ? $\frac{2}{3}$?
4. Represent $7 \cdot \frac{1}{6}$ using segments of equal length on the number line.
5. Write each of the following as indicated products.

(a) $\frac{4}{5}$

(c) $-\frac{9}{15}$

(e) $-\frac{19}{17}$

(b) $\frac{3}{7}$

(d) $\frac{21}{5}$

(f) $-\frac{22}{53}$

6-4. Reciprocals

Definition: Two numbers are said to be the reciprocals of each other if their product is equal to 1.

You know from the previous section that

$$1 = \frac{2}{2} = 2 \cdot \frac{1}{2}; \quad 1 = \frac{4}{4} = \frac{1}{4} \cdot 4; \quad 1 = \frac{17}{17} = 17 \cdot \frac{1}{17}.$$

Our definition of reciprocal tells us that

$\frac{1}{2}$ is the reciprocal of 2,

4 is the reciprocal of $\frac{1}{4}$, and

$\frac{1}{17}$ is the reciprocal of 17,

because in each case the product of these pairs of numbers is 1. Since the number 1 can always be expressed in the form $\frac{a}{a}$ ($a \neq 0$), we can then say that if $a \neq 0$

$$1 = \frac{a}{a} = a \cdot \frac{1}{a}.$$

The number $\frac{1}{a}$ is called the reciprocal of a . Also, a is the reciprocal of $\frac{1}{a}$.

You have seen that some rational numbers have reciprocals. What about rational numbers of the form $\frac{a}{b}$? For example,

$$\frac{5}{8}, \frac{7}{5}, \frac{9}{10}, \text{ etc.};$$

do these rational numbers have reciprocals? Let us consider the rational number $\frac{5}{8}$ and see if we can find another rational number such that the product of the two is 1, that is

$$\frac{5}{8} \cdot (?) = 1.$$

We know that $\frac{5}{8} = 5 \cdot \frac{1}{8}$. Also that

$$5 \cdot \left(\frac{1}{5}\right) = 1 \text{ and } \frac{1}{8} \cdot (8) = 1.$$

Since you know that

$$1 \cdot 1 = 1,$$

it must follow that

$$\left(5 \cdot \left(\frac{1}{5}\right)\right) \cdot \left(\frac{1}{8} \cdot (8)\right) = 1.$$

By making use of the commutative and associative properties of multiplication, we can rewrite this sentence as

$$\left(5 \cdot \frac{1}{8}\right) \cdot \left(\frac{1}{5} \cdot 8\right) = \frac{5}{8} \cdot \frac{8}{5} = 1.$$

We can see then that the reciprocal of $\frac{5}{8}$ is $\frac{8}{5}$ because

$$\frac{5}{8} \cdot \frac{8}{5} = 1.$$

In general:

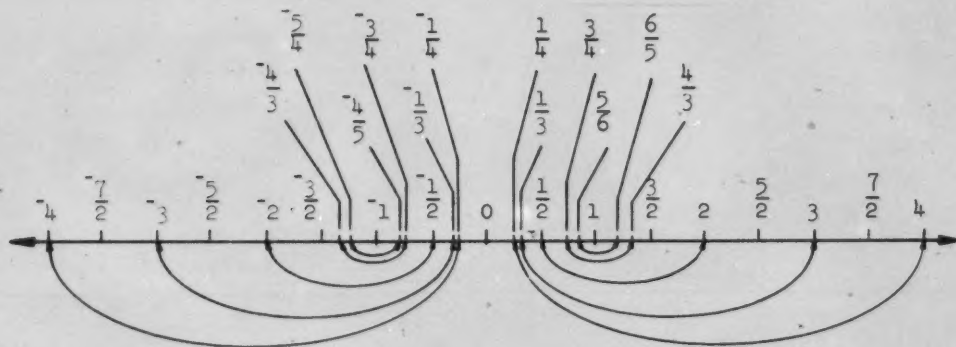
If $a \neq 0$ and $b \neq 0$, then the reciprocal of the rational number $\frac{a}{b}$ is the rational number $\frac{b}{a}$. That is,

$$\begin{aligned}
 \frac{a}{b} \cdot \frac{b}{a} &= a \cdot \frac{1}{b} \cdot b \cdot \frac{1}{a} \\
 &= a \cdot \frac{1}{a} \cdot b \cdot \frac{1}{b} \\
 &= \frac{a}{a} \cdot \frac{b}{b} \\
 &= 1 \cdot 1 \\
 &= 1
 \end{aligned}$$

Exercises 6-4a

(Class Discussion)

On the diagram below, some numbers and their reciprocals are joined.



1. Notice the pattern in the diagram above. If we locate a rational number very close to 0, where will the location of its reciprocal be?
2. How can you test to see if these pairs of numbers really are the reciprocals of each other?
3. What does the pattern of the diagram above indicate about the reciprocal of 0?
4. What is the reciprocal of 1? of opp 1?

Exercises 6-4b

1. Find the rational number x that will make the following sentences true.

(a) $3 \cdot x = 1$	(g) $\frac{5}{6} \cdot x = 1$
(b) $\frac{1}{2} \cdot x = 1$	(h) $(-\frac{3}{7}) \cdot x = 1$
(c) $(-3) \cdot x = 1$	(i) $x \cdot -7 = 1$
(d) $(-\frac{1}{2}) \cdot x = 1$	(j) $\frac{3}{10} \cdot x = 1$
(e) $x \cdot \frac{3}{4} = 1$	(k) $x \cdot \frac{1}{100} = 1$
(f) $x \cdot 7 = 1$	() $x \cdot 100 = 1$
 2. Draw a number line and mark off with double arrows the rational numbers 3 , $\frac{1}{3}$, -3 , $-\frac{1}{2}$, $\frac{3}{4}$, 7 , and their reciprocals.
 3. Draw a number line and mark off with double arrows the rational numbers 3 , $\frac{1}{3}$, -3 , $-\frac{1}{2}$, $\frac{3}{4}$, 7 , and their opposites. How does the pattern of the double arrows differ from the pattern in Problem 2?
 4. Suppose that two rational numbers are the reciprocals of each other and one number is < 0 , then the other number must be ____? ____.
-

6-5. New Ways of Naming a Number

In our work with numbers so far, we have found it convenient to use an upper dash to refer to a negative number. For example, if we wish to express negative eight we write $"-8"$. Notice that if the dash is in this raised position, it is not likely that a beginning student will confuse it with the operational symbol $(-)$ for subtraction. Now that you

- (1) are familiar with negative numbers, and
- (2) will not confuse their names with subtraction,

let's agree to use a lower dash to refer to any of them. That is, now we will express negative eight by writing $"-8"$, and we may think of the

lower dash as meaning either "negative" or "the opposite of". Therefore, "-8" may be read as either

- (1) negative eight, or
- (2) the opposite of eight.

In terms of our new information we can now express subtraction as follows:

If a and b are any two rational numbers,
then " $a - b = a + (-b)$ ".

That is, "a subtract b equals a plus the opposite of b". Note that b may represent a positive number, zero, or a negative number. Therefore, "-b" must be read as either "the negative of b" or "the opposite of b".

At this time, let's see if we can discover anything about the three fractions $-\frac{15}{3}$, $\frac{-15}{3}$, $\frac{15}{-3}$.

Study the following:

$$-\frac{15}{3} = -(15 \cdot \frac{1}{3}) = -5,$$

$$\frac{-15}{3} = (-15) \cdot \frac{1}{3} = -5,$$

$$\frac{15}{-3} = 15 \cdot (-\frac{1}{3}) = -5.$$

The above work indicates that $-\frac{15}{3}$, $\frac{-15}{3}$, and $\frac{15}{-3}$ are merely different names for (-5)!

Now, let's examine the fractions $\frac{15}{3}$ and $\frac{-15}{-3}$.

$$\frac{15}{3} = 15 \cdot \frac{1}{3} = 5$$

$$\frac{-15}{-3} = (-15) \cdot (-\frac{1}{3}) = 5$$

From our work with fractions in the last two examples, we can now state that

If $b \neq 0$, then

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{b}, \text{ and}$$

$$\frac{a}{b} = \frac{-a}{-b}.$$

Exercises 6-5

1. What kind of a number is $-x$ if
 - (a) x is positive?
 - (b) x is a negative number?
 - (c) x is zero?
2. What kind of a number is x if
 - (a) $-x$ is positive?
 - (b) $-x$ is a negative number?
 - (c) $-x$ is zero?
3. Form the opposite of each of the following numbers.

(a) $-(-2)$	(c) $-(42 \cdot 0)$	(e) $-(15 \cdot 1)$
(b) $-(5 + (-7))$	(d) $-(42 + 0)$	(f) $-(15 \cdot (-1))$
4.
 - (a) Is every rational number the opposite of some rational number?
 - (b) Is every opposite of a number a negative number?
5. Write common names for the following.

(a) $\frac{75}{5}$	(e) $\frac{(-5)}{(-5)}$
(b) $\frac{570}{570}$	(f) $\frac{(-45)}{15}$
(c) $\frac{(-30)}{(-5)}$	(g) $\frac{(-1)}{(-1)}$
(d) $\frac{(-30)}{5}$	(h) $\frac{2500}{(-2)}$
6. Brainbooster! If $\frac{a}{b} = a \cdot \frac{1}{b}$, $b \neq 0$, what is

$$\frac{\frac{2}{3}}{\frac{2}{3}} ?$$

6-6. Multiplication of Rational Numbers

From your work in the last sections you know the reciprocal of $7 = \frac{1}{7}$ and the reciprocal of $5 = \frac{1}{5}$. Now, we would like to know what the product of two rational numbers such as $\frac{1}{7}$ and $\frac{1}{5}$ would be, that is

$$\frac{1}{7} \cdot \frac{1}{5} = (?).$$

Consider the following:

$$\begin{aligned}(35) \cdot \left(\frac{1}{7} \cdot \frac{1}{5}\right) &= (7 \cdot 5) \cdot \left(\frac{1}{7} \cdot \frac{1}{5}\right) \text{ (renaming 35 as } 7 \cdot 5\text{)} \\ &= (7 \cdot \frac{1}{7}) \cdot (5 \cdot \frac{1}{5}) \text{ (using the commutative and associative properties)} \\ &= 1 \cdot 1 \text{ (definition of reciprocal)} \\ &= 1.\end{aligned}$$

We know that

$$\frac{35}{35} = (35) \cdot \left(\frac{1}{35}\right) = 1,$$

and we also know from above that

$$(35) \cdot \left(\frac{1}{7} \cdot \frac{1}{5}\right) = 1.$$

Hence,

$$\frac{1}{7} \cdot \frac{1}{5} = \frac{1}{35}.$$

This result now provides us with the means of finding the product of any two rational numbers. For example:

$$\begin{aligned}\frac{2}{3} \cdot \frac{5}{7} &= 2 \cdot \left(\frac{1}{3}\right) \cdot 5 \cdot \left(\frac{1}{7}\right) \\ &= (2 \cdot 5) \cdot \left(\frac{1}{3} \cdot \frac{1}{7}\right) \\ &= (2 \cdot 5) \cdot \left(\frac{1}{3 \cdot 7}\right) \\ &= (10) \cdot \left(\frac{1}{21}\right) \\ &= \frac{10}{21}.\end{aligned}$$

Of course, you are not expected to go through this procedure each time you multiply two rational numbers. It can be seen from the example above that

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}.$$

To multiply two rational numbers written in fraction form, find the fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators. (Notice that this tells you the product of two rational numbers is a single rational number. This suggests that the set of rational numbers is closed under the operation of multiplication.)

In symbols, if $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers,

$$(1) \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}.$$

It is also true that

$$(2) \frac{ac}{bd} = \frac{a \cdot c}{b \cdot d} = \frac{a}{b} \cdot \frac{c}{d}.$$

Exercises 6-6a

(Class Discussion)

1. (a) What is the reciprocal of 8?
 (b) What is the reciprocal of 9?
 (c) What is the reciprocal of (8 · 9)?
2. If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$.

Use this information to do the following multiplication. Do not leave out a step.

$$(a) \frac{2}{3} \cdot \frac{5}{7}$$

$$(c) \frac{1}{3} \cdot \frac{5}{6}$$

$$(b) \frac{3}{4} \cdot \frac{9}{8}$$

$$(d) \frac{4}{9} \cdot \frac{5}{11}$$

3. If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{ac}{bd} = \frac{a \cdot c}{b \cdot d} = \frac{a}{b} \cdot \frac{c}{d}$.

Use this information to find the two rational numbers whose product is listed below. Do not leave out a step.

(a) $\frac{10}{33}$

(c) $\frac{7}{121}$

(b) $\frac{6}{21}$

(d) $\frac{35}{39}$

We can use the statement, if $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{ac}{bd} = \frac{a \cdot c}{b \cdot d} = \frac{a}{b} \cdot \frac{c}{d}$, to find the "simplest" name for a rational number.

Before we go on, recall that a prime number is a positive integer that has exactly two different factors. For example, 5 is prime because 1 and 5 are the only factors of 5 that are integers. In the same manner, 17 is prime because 1 and 17 are the only factors of 17 that are integers. With the exception of 1, all positive integers that are not prime are called composite numbers. Any composite number can be expressed as a product of prime factors.

Now, if we wish to find the "simplest" name for a rational number, say $\frac{6}{10}$, we can proceed as follows:

$$6 = 2 \cdot 3 \quad \text{and} \quad 10 = 2 \cdot 5.$$

Therefore,

$$\frac{6}{10} = \frac{2 \cdot 3}{2 \cdot 5} = \frac{2}{2} \cdot \frac{3}{5} = 1 \cdot \frac{3}{5} = \frac{3}{5}.$$

In the same manner:

$$\frac{30}{36} = \frac{5 \cdot 6}{4 \cdot 9} = \frac{5 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 3} = \frac{2}{2} \cdot \frac{3}{3} \cdot \frac{5}{2 \cdot 3} = 1 \cdot 1 \cdot \frac{5}{6} = \frac{5}{6}.$$

It should be noted that $\frac{30}{36}$ and $\frac{5}{6}$ are both acceptable names for the same number, but we often prefer the "simplest" name because it frequently enables us to compute more easily.

We are not going to consider the integral factors of a negative integer. You will recall that any negative integer can be expressed as the product of (-1) and that integer, thus,

$$-12 = (-1) \cdot 12.$$

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Therefore the need to consider the integral factors of a negative integer is eliminated because

$$-12 = (-1) \cdot 12 = (-1) \cdot 2 \cdot 2 \cdot 3.$$

Exercises 6-6b

(Class Discussion)

1. Write each of the following as the product of prime factors.

Example: $36 = 4 \cdot 9 = 2 \cdot 2 \cdot 3 \cdot 3$

- | | |
|-----------|------------|
| (a) 10 | (e) 45 |
| (b) (-15) | (f) (-50) |
| (c) 9 | (g) 19 |
| (d) 30 | (h) (-100) |
2. Write a mathematical sentence indicating that "3 is a factor of 12".
3. Write a mathematical sentence indicating that "35 is a multiple of 7".
4. The integer 36 has nine integral factors. Find them!
5. If an integer has 2 as one of its factors it is an even integer. If an integer does not have 2 as one of its factors it is an odd integer. Tell whether these numbers are odd or even.
- | | |
|----------------------------|----------------------------|
| (a) $2 \cdot 5$ | (f) $3 \cdot 2 \cdot 6$ |
| (b) $3 + 7$ | (g) $126 - 38$ |
| (c) $(-6) \cdot 5 \cdot 3$ | (h) $3 \cdot (-3) \cdot 7$ |
| (d) $(-2) + 16$ | (i) $7 + (-7)$ |
| (e) $7 + 8$ | (j) 9,573,175,937,997,552 |
6. (a) Can a prime integer be even?
- (b) How many prime integers end with the digit 5?
- (c) Why do we not consider 1 a prime integer?

7. Find the "simplest" name for each of the following by expressing numerator and denominator as products in prime factored form and then simplify, using the multiplication property of 1.

Example:

$$\frac{18}{24} = \frac{2 \cdot 9}{4 \cdot 6} = \frac{2 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 3} = \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{3}{2 \cdot 2} = 1 \cdot 1 \cdot \frac{3}{4} = \frac{3}{4}$$

- | | |
|---------------------|---------------------|
| (a) $\frac{15}{25}$ | (d) $\frac{8}{9}$ |
| (b) $\frac{12}{16}$ | (e) $\frac{34}{51}$ |
| (c) $\frac{22}{33}$ | (f) $\frac{36}{20}$ |

Exercises 6-6c

1. Find the following products. Write your answers in simplest form.

- | | | |
|---|--|---|
| (a) $\frac{1}{2} \cdot \frac{1}{3}$ | (i) $(-\frac{1}{5}) \cdot \frac{2}{3}$ | (q) $\frac{3}{4} \cdot \frac{1}{5}$ |
| (b) $\frac{2}{5} \cdot \frac{3}{5}$ | (j) $\frac{1}{10} \cdot \frac{3}{10}$ | (r) $\frac{4}{100} \cdot \frac{3}{10}$ |
| (c) $\frac{1}{9} \cdot 9$ | (k) $(-\frac{2}{9}) \cdot (-2)$ | (s) $2 \cdot \frac{2}{9}$ |
| (d) $4 \cdot (-\frac{1}{3})$ | (l) $\frac{1}{3} \cdot \frac{1}{3}$ | (t) $\frac{3}{2} \cdot 4$ |
| (e) $8 \cdot \frac{3}{4}$ | (m) $(\frac{1}{2} \cdot \frac{1}{4}) \cdot (-\frac{1}{3})$ | (u) $(-\frac{1}{10}) \cdot \frac{1}{10} \cdot (-\frac{1}{10})$ |
| (f) $(-10) \cdot \frac{1}{100}$ | (n) $\frac{1}{100} \cdot \frac{1}{100}$ | (v) $\frac{1}{10} \cdot \frac{1}{100}$ |
| (g) $\frac{1}{10} \cdot 100$ | (o) $\frac{5}{2} \cdot (-8)$ | (w) $\frac{5}{6} \cdot \frac{3}{3} \cdot \frac{1}{5}$ |
| (h) $(-\frac{6}{5}) \cdot (-\frac{5}{6})$ | (p) $(-\frac{21}{2}) \cdot \frac{1}{2}$ | (x) $(-\frac{1}{2}) \cdot (-\frac{1}{2}) \cdot (-\frac{1}{2}) \cdot (-\frac{1}{2})$ |

2. Find the products in simplest form.

- | | | |
|---|--|---|
| (a) $\frac{1}{3} \cdot \frac{1}{13}$ | (f) $(-\frac{1}{8}) \cdot (-\frac{1}{15})$ | (k) $\frac{1}{12} \cdot (-\frac{1}{5})$ |
| (b) $\frac{1}{5} \cdot 50$ | (g) $\frac{2}{3} \cdot \frac{1}{8}$ | (l) $(-\frac{5}{6}) \cdot (-24)$ |
| (c) $(-36) \cdot (-\frac{1}{12})$ | (h) $(-\frac{4}{7}) \cdot (-\frac{1}{12})$ | (m) $(-72) \cdot \frac{5}{9}$ |
| (d) $\frac{8}{9} \cdot (-1)$ | (i) $\frac{4}{5} \cdot \frac{17}{17}$ | (n) $\frac{9}{16} \cdot \frac{7}{10}$ |
| (e) $\frac{13}{15} \cdot \frac{10}{39}$ | (j) $\frac{33}{100} \cdot \frac{90}{99}$ | (o) $(-\frac{17}{3}) \cdot (-\frac{3}{51})$ |

3. Often factoring may make the computation of products like those in Problem 2 easier. For example, in part (g)

$$\frac{2}{3} \cdot \frac{1}{8} = \frac{2 \cdot 1}{3 \cdot 8} = \frac{2 \cdot 1}{3 \cdot (2 \cdot 4)} = \frac{2 \cdot 1}{2 \cdot 3 \cdot 4} = \frac{2}{2} \cdot \frac{1}{3 \cdot 4} = 1 \cdot \frac{1}{12} = \frac{1}{12}.$$

Use this method for parts e, h, j, and o of Problem 2.

4. We know that if $b \neq 0$, then $\frac{a}{b} = a \cdot \frac{1}{b}$ (read "a times the reciprocal of b"). Use this knowledge to find answers to the following problems. Write your answers in simplest form.

(a) $\frac{4}{\frac{1}{2}}$ (c) $\frac{\frac{1}{2}}{4}$ (e) $\frac{\frac{2}{3}}{\frac{1}{2}}$

(b) $\frac{1}{4 \frac{1}{4}}$ (d) $\frac{\frac{1}{5}}{2 \frac{1}{10}}$ (f) $\frac{2 \frac{1}{3}}{3 \frac{1}{4}}$

5. Find each product as a fraction in simplest form. Before you multiply, replace each mixed numeral with a single fraction. For example,

$$\frac{3}{4} \cdot (-2 \frac{1}{2}) = \frac{3}{4} \cdot (-\frac{5}{2}) = (-\frac{15}{8}).$$

(a) $(-8) \cdot (2 \frac{2}{3})$ (d) $(-3 \frac{1}{3}) \cdot (-2 \frac{2}{3})$

(b) $12 \cdot (5 \frac{5}{6})$ (e) $(2 \frac{1}{5}) \cdot (-3 \frac{1}{3})$

(c) $(5 \frac{1}{4}) \cdot (\frac{2}{7})$ (f) $(1 \frac{1}{7}) \cdot (1 \frac{1}{8})$

6. Find the replacement for r which will make the following sentence true.

$$\frac{3}{4} \cdot (\frac{1}{3} \cdot \frac{7}{8}) = (r \cdot \frac{1}{3}) \cdot \frac{7}{8}.$$

7. The label on a can of paint states that the can contains $\frac{3}{4}$ of a pint. If you use $\frac{2}{3}$ of the paint, how many pints did you use?

8. Here is a recipe for fruit punch to serve 6 persons:

$\frac{1}{2}$ cup orange juice

$\frac{1}{4}$ cup lemon juice

- $\frac{1}{2}$ cup grapefruit juice
- 1 cup pineapple juice
- $1\frac{1}{2}$ cups water
- $\frac{1}{2}$ cup syrup

- (a) What amount of each ingredient would you use to make punch for 4 persons?
 - (b) What amount of each ingredient would you use to serve 8 persons?
9. On a road map 1 inch represents 10 miles. If the distance on this map from your house to school measures $1\frac{5}{8}$ inches, how many miles do you live from school?
-

6-7. Addition of Rational Numbers

Let us consider the problem of finding a single fraction for the rational number

$$\left(\frac{2}{3} + \frac{5}{7}\right)$$

and see if we can call upon what we have learned so far to help us find this fraction.

We know that 21 is a multiple of both 3 and 7, the denominators. We also know that

$$21 \cdot \frac{2}{3} = 14$$

and

$$21 \cdot \frac{5}{7} = 15.$$

It appears then, that if we could somehow multiply

$$\left(\frac{2}{3} + \frac{5}{7}\right)$$

by 21, we could then relate the addition of these numbers to that of adding integers; which we know how to do! Our multiplication property of 1 provides us with the means of performing this multiplication since

multiplying a number by 1 leaves that number unchanged, that is

$$1 \cdot \left(\frac{2}{3} + \frac{5}{7}\right) = \frac{21}{21} \cdot \left(\frac{2}{3} + \frac{5}{7}\right) = \left(\frac{2}{3} + \frac{5}{7}\right).$$

We also know that $\frac{21}{21}$ can be renamed as $\frac{1}{21} \cdot 21$.

Our knowledge of the multiplication property of 1 and our ability to use the distributive property now permits us to do the addition.

$$\begin{aligned}\frac{2}{3} + \frac{5}{7} &= \frac{21}{21} \cdot \left(\frac{2}{3} + \frac{5}{7}\right) \\ &= \frac{1}{21} \cdot 21 \left(\frac{2}{3} + \frac{5}{7}\right) \\ &= \frac{1}{21} \cdot \left(\frac{21 \cdot 2}{3} + \frac{21 \cdot 5}{7}\right) \\ &= \frac{1}{21} \cdot (7 \cdot 2 + 3 \cdot 5) \\ &= \frac{1}{21} \cdot (14 + 15) \\ &= \frac{1}{21} \cdot (29) \\ &= \frac{29}{21}.\end{aligned}$$

Look at one more example.

$$\begin{aligned}\frac{3}{4} - \frac{7}{10} &= \frac{3}{4} + \left(-\frac{7}{10}\right) = \frac{3}{4} + \frac{(-7)}{10} \\ \frac{3}{4} + \frac{(-7)}{10} &= \frac{20}{20} \cdot \left(\frac{3}{4} + \frac{(-7)}{10}\right) \\ &= \frac{1}{20} \cdot 20 \left(\frac{3}{4} + \frac{(-7)}{10}\right) \\ &= \frac{1}{20} \cdot \left(\frac{20 \cdot 3}{4} + \frac{20 \cdot (-7)}{10}\right) \\ &= \frac{1}{20} \cdot (5 \cdot 3 + 2 \cdot (-7)) \\ &= \frac{1}{20} \cdot (15 + (-14)) \\ &= \frac{1}{20} \cdot (1) \\ &= \frac{1}{20}.\end{aligned}$$

Exercises 6-7a

(Class Discussion)

1. In each case, use the distributive property to find the sums.

(a) $\frac{2}{3} + \frac{4}{3}$

(d) $\frac{12}{8} + \frac{-3}{8}$

(b) $\frac{-4}{3} + \frac{2}{3}$

(e) $\frac{12}{8} - \frac{3}{8}$

(c) $-\frac{3}{8} + \frac{12}{8}$

(f) $\frac{3}{8} + \frac{-12}{8}$

2. Find the least common multiple for each of the following pairs of integers.

(a) 6, 8

(d) 12, 18

(b) 4, 12

(e) 15, 25

(c) 9, 21

(f) 10, 100

3. Using your results from Problem 2, perform the following additions.

(a) $\frac{5}{6} + \frac{5}{8}$

(d) $\frac{11}{12} + \frac{1}{-18}$

(b) $\frac{3}{4} + \frac{-7}{12}$

(e) $\frac{-13}{15} + \frac{-11}{25}$

(c) $\frac{5}{9} + \frac{-10}{21}$

(f) $\frac{7}{-10} + \frac{-1}{100}$

4. Find the following sums. Use the associative and commutative properties to make your work easier.

(a) $5\frac{5}{6} + (3\frac{4}{6})$

(b) $6\frac{7}{8} + (-4\frac{3}{4})$

Exercises 6-7b

1. Find the sum of each of the following. Give answers in simplest form.

(a) $\frac{73}{64} + \frac{20}{32}$

(e) $-\frac{3}{8} + \frac{5}{7}$

(b) $\frac{10}{21} + -\frac{2}{3}$

(f) $-\frac{5}{12} + -\frac{7}{8}$

(c) $(\frac{1}{3} \cdot \frac{1}{2}) + (\frac{1}{3} \cdot \frac{1}{5})$

(g) $-\frac{2}{3} + -\frac{5}{6}$

(d) $\frac{1}{3} \cdot (\frac{1}{2} + \frac{1}{5})$

(h) $\frac{7}{8} + (-\frac{6}{8})$

2. Find each of the following sums. Use the commutative and associative properties to make your work easier.

Example: $1\frac{1}{3} + 2\frac{2}{3} = (1 + \frac{1}{3}) + (2 + \frac{2}{3}) = (1 + 2) + (\frac{1}{3} + \frac{2}{3})$
 $= 3 + (1) = 4.$

(a) $33\frac{1}{3} + 66\frac{2}{3}$

(c) $30 + 7\frac{4}{5}$

(b) $16\frac{1}{2} + 37\frac{1}{3}$

(d) $2\frac{1}{4} + 3\frac{5}{8}$

3. Find each of the following sums.

(a) $4\frac{4}{8} + (-3\frac{1}{8})$

(c) $2\frac{12}{32} - (-14\frac{23}{32})$

(b) $8\frac{5}{16} + (-1\frac{13}{16})$

(d) $3 + (-\frac{2}{3})$

(Hint: $3 = 2 + 1 = 2 + \frac{2}{3}$)

6-8. Ratios and Percents Expressed by Rational Numbers

In your use of numbers, you probably noticed that you can compare two numbers by division. Of the two numbers, 6 and 2, we can say that the first number is three times the second. We can also say that the ratio of the first number to the second is 3 to 1 or 3. In comparing the numbers 9 and 2, we sometimes say that the ratio of these numbers is 9 to 2 or $\frac{9}{2}$. Nine is four and one-half times the number 2.

Definition: The ratio of a number c to a number d , $d \neq 0$, is $\frac{c}{d}$.

In a class there are 36 pupils of whom 16 are girls. The ratio of the number of girls in the class to the number of class members is $\frac{16}{36}$ or $\frac{4}{9}$.

Thus $\frac{4}{9}$ of the class members are girls: $36 \cdot (\frac{4}{9}) = 16$. The ratio of the number of pupils in the class to the number of girls in the class is $\frac{36}{16}$ or $\frac{9}{4}$. There are two and one-fourth times as many pupils as girls in the class. Notice that $\frac{9}{4} = \frac{8}{4} + \frac{1}{4} = 2 + \frac{1}{4}$.

Ratio is very useful in many applications of mathematics. In the remainder of the work, this year as well as next, we will see many other examples of ratios.

Exercises 6-8a

(Class Discussion)

1. Express the ratios of the following as rational numbers in simplest form:
 - (a) 3 feet to 10 feet.
 - (b) A test grade of 75 to a test grade of 90.
 - (c) 56 pounds to 12 pounds.
 - (d) $\frac{9}{2}$ to $\frac{7}{2}$.
 - (e) $\frac{5}{4}$ inches to 2 inches.
 - (f) 90 cents to 75 cents.
 - (g) 15,000 people to 25,400 people.
 - (h) 90 degrees to 55 degrees.

2.

x	.1	2	3	4	5	6
3x	3	6	9	12	15	18

Find the ratio of each number in the second line of the table to the corresponding number in the first line of the table.

3.

2	4	5	8	10		
5	10	$12\frac{1}{2}$	20		30	40

Assume the ratio of each pair of corresponding numbers (numbers in the same vertical column) in the table is the same. What are the missing numbers?

Many of you are familiar with the word "percent", and you may know something about its meaning. If your teacher says, "90 percent of the answers on this paper are correct", would you know what he means? The word "percent" comes from the Latin phrase "per centum", which means "by the hundred". If the paper with 90 of the answers correct has 100 answers, then 90 answers out of the 100 are correct. The ratio $\frac{90}{100}$ could be used instead of the phrase "90 percent" to describe the part of the answers which are correct. The word "percent" is used when a ratio is expressed with a denominator of 100.

$$90 \text{ percent} = \frac{90}{100} = 90 \cdot \frac{1}{100}.$$

For convenience the symbol, %, is used for the word "percent". This symbol is a short way of saying $\frac{1}{100}$.

$$\frac{90}{100} = 90 \cdot \frac{1}{100} = 90\%.$$

$$\frac{16}{100} = 16 \cdot \frac{1}{100} = 16\%.$$

$$\frac{37}{100} = 37 \cdot \frac{1}{100} = 37\%.$$

$$\frac{77}{100} = 77 \cdot \frac{1}{100} = 77\%.$$

$$? = 13 \cdot \frac{1}{100} = 13\%.$$

Suppose that the paper has 90 correct answers out of the 100; 6 incorrect answers out of the 100; 4 answers omitted out of the 100. Since we know the total number of problems we can express this information in terms of percent.

$$\frac{90}{100} = 90 \cdot \frac{1}{100} = 90\% \quad (90\% \text{ of the problems were correct.})$$

$$\frac{6}{100} = 6 \cdot \frac{1}{100} = 6\% \quad (6\% \text{ of the problems were incorrect.})$$

$$\frac{4}{100} = 4 \cdot \frac{1}{100} = 4\% \quad (4\% \text{ of the problems were omitted.})$$

$$\frac{90}{100} + \frac{6}{100} + \frac{4}{100} = \frac{100}{100} \quad (\text{all possible answers.})$$

$$90\% + 6\% + 4\% = 100\% \quad (\text{all possible answers.})$$

Another name for the number one is 100%.

The number 2 can be written

$$\frac{2}{1} = \frac{2}{1} \cdot \frac{100}{100} = \frac{200}{100} = 200\%.$$

In other words, 200% means $200 \cdot \frac{1}{100} = \frac{200}{100} = 2$.

A class of 25 pupils is made up of 11 girls and 14 boys. The ratio of the number of girls to the number of pupils in the class can be expressed many ways. For instance:

$$1 \cdot \frac{11}{25} = \frac{22}{50} = \frac{33}{75} = \frac{44}{100} = \frac{55}{125} = \frac{66}{150}.$$

If we wish to indicate the percent of the class that is girls, which fraction gives the information most easily? Why? The ratio of the number of boys to the total number in the class may be written

$$1 \cdot \frac{14}{25} = \frac{c}{50} = \frac{d}{75} = \frac{56}{100} = \frac{e}{125} = \frac{f}{150}.$$

What numbers are represented by the letters c, d, e, f? Notice the two ratios $\frac{11}{25}$ (girls) and $\frac{14}{25}$ (boys). What is the sum of the two ratios? Find the sum of the two ratios $\frac{44}{100}$ and $\frac{56}{100}$. Express the two ratios and their sum as percents using the symbol, %. The entire class is considered to be 100% of the class because

$$\frac{25}{25} = \frac{100}{100} = 100\%.$$

Any number $\frac{a}{b}$ can be expressed as a percent by finding the number c such that

$$\frac{a}{b} = \frac{c}{100} = c \cdot \frac{1}{100} = c\%.$$

Percent is a convenient tool for giving information involving ratios. Athletic standings sometimes are given in percent. Two seventh-grade pupils discovered the reason for this use of percent. The boys were discussing the scores of their baseball teams. In the Little League one team won 15 games out of 20 games played. Another team won 18 out of 25 games. Which team had a better record? The second team had won 3 more games, but the first team played fewer games. Look at the ratios of the number of games won to the number of games played for each team. The ratios $\frac{15}{20}$ and $\frac{18}{25}$ cannot be compared at a glance. Let us use percent for the comparison.

The first team won $\frac{15}{20}$ of the games played. $\frac{5}{5} \cdot \frac{15}{20} = \frac{75}{100} = 75\%$. They won 75% of the games played.

The second team won $\frac{18}{25}$ of the games played. $\frac{4}{4} \cdot \frac{18}{25} = \frac{72}{100} = 72\%$. They won 72% of the games played.

The first team which won 75% of its games had a higher standing than the second team which won 72% of its games. We could say that $72\% < 75\%$ or $75\% > 72\%$.

Data about business, school, Scouts--are sometimes given in percent. It is often more convenient to refer to the data at some later time if they are given in percent than if they are given otherwise.

A few years ago the director of a camp kept some records for future use. Some information was given in percent, and some was not. The records gave the following items of information.

- (1) There were 200 boys in camp.
- (2) One hundred percent of the boys were hungry for the first dinner in camp.
- (3) On the second day in camp 44 boys caught fish.
- (4) One boy wanted to go home the first night.
- (5) A neighboring camp director said "forty percent of the boys in my camp will learn to swim this summer. We shall teach 32 boys to swim."

From (1) and (2), how many hungry boys came to dinner the first day?

$$\frac{100}{100} \cdot 200 = 200$$

Of course we should know without computation that 100% of 200 is 200.

From (3), we can find the percent of boys who caught fish on the second day. The ratio of the number of boys who caught fish to the number of boys in camp is $\frac{44}{200}$. If we call $x\%$ the percent of the boys who caught fish, then

$$\frac{x}{100} = \frac{44}{200}$$

$$\frac{x}{100} = \frac{2 \cdot 22}{2 \cdot 100}$$

$$\frac{x}{100} = \frac{22}{100}$$

The input for x which will make this a true statement is 22; thus twenty-two percent of the boys caught fish.

From (4), we can find the percent of the boys who were homesick. If this is $x\%$, then

$$\frac{x}{100} = \frac{1}{200}$$

$$x \cdot \frac{1}{100} = \frac{1}{2 \cdot 100}$$

$$x \cdot \frac{1}{100} = \frac{1}{2} \cdot \frac{1}{100}$$

The input for x which will make this a true statement is $\frac{1}{2}$; thus, of the boys, $\frac{1}{2}\%$ were homesick. This may be read "one-half percent" or "one half of one percent". You may prefer to say "one half of one percent", since this emphasizes the smallness of it.

From the information in Item 5, the total number of boys in the second camp can be found. Call this number of boys x . 40% means $\frac{40}{100}$ of the group, and also refers to 32 boys. We wish to find x such that $\frac{40}{100} = \frac{32}{x}$.

$$\frac{40}{100} = \frac{32}{x}$$

$$\frac{4 \cdot 10}{10 \cdot 10} = \frac{32}{x}$$

$$\frac{4}{10} = \frac{32}{x}$$

$$\frac{8}{8} \cdot \frac{4}{10} = \frac{32}{x}$$

$$\frac{32}{80} = \frac{32}{x}$$

The input for x which will make this a true statement is 80.

Exercises 6-8b

1. A Little League team won 3 out of the first 5 games played.
 - (a) What percent of the first 5 games did the team win?
 - (b) What percent of the first 5 games did the team lose?
2. Later in the season the team had won 8 out of the 16 games played.
 - (a) What was the percent of games won at this time?
 - (b) Has the percent of games won increased or decreased?
3. At the end of the season, the team had won 26 games out of 40.
 - (a) What percent of the games played did the team win by the end of the season?
 - (b) How does this percent compare with the other two?
4. There are 600 seventh-grade pupils in a junior high school. The principal hopes to divide the pupils into 20 sections of equal size.
 - (a) How many pupils would be in each section?
 - (b) What percent of the pupils would be in each section?
 - (c) What number of pupils is 1% of the number of pupils in the seventh grade?
 - (d) What number of pupils is 10% of the number of pupils in the seventh grade?
5. Suppose one section contains 36 pupils. What percent of the seventh-grade pupils are in that section?
6. One hundred fifty seventh-grade pupils come to school on the school bus.
 - (a) What percent of the seventh-grade pupils come by school bus?
 - (b) What percent of the seventh-grade pupils come to school by some other means?
7. In a section of 30 pupils, 3 were tardy one day.
 - (a) What fractional part of that section was tardy?
 - (b) What percent of that section was tardy?

8. There were 750 eighth-grade pupils. The number of eighth-grade pupils is what percent of the number of seventh-grade pupils?
 9. One day 3 seventh-grade pupils came to school on crutches (they had been skiing). What percent of the number of seventh-grade pupils were on crutches?
-

6-9. Decimal Notation

You recall that a base ten numeral, such as 3284, written as $3 \cdot (10^3) + 2 \cdot (10^2) + 8 \cdot (10^1) + 4$ was said to be in expanded form. Written as 3284 the numeral is said to be in positional notation. In base ten this form is also called decimal notation. Each digit represents a certain value according to its place in the numeral. In the above example, the 3 is in thousands place, the 2 is in hundreds place and so on. The form for place value in base ten shows that the value of each place immediately to the left of a given place is ten times the value of the given place. Each place value immediately to the right of a given place is one tenth of the place value of the given place. Looking back to the number 3284 written in expanded form you will notice that reading from left to right the exponents of ten decrease. Suppose we want to write 5634.728 in expanded form. We could write

$5 \cdot (10^3) + 6 \cdot (10^2) + 3 \cdot (10^1) + 4 + 7 \cdot (?) + 2 \cdot (?) + 8 \cdot (?)$. The 4 is in unit's (or one's) place. The value of this place is $\frac{1}{10}$ of the value of the place before it. If we extend our numeral to the right of one's place and still keep this pattern, what will be the value of the next place? Of the next place after that one? To write this in expanded form to the right of unit's place as well as to the left of unit's place we have:

$5 \cdot (10^3) + 6 \cdot (10^2) + 3 \cdot (10^1) + 4 + 7 \cdot (\frac{1}{10}) + 2 \cdot (\frac{1}{10^2}) + 8 \cdot (\frac{1}{10^3})$ where the first place to the right of one's place is the one-tenth's place or simply tenth's place.

The following chart shows the place values both to the left and to the right of the unit's place.

Place Value Chart

Hundred-thousand	Ten-thousand	Thousand	Hundred	Ten	Unit or one	Tenth	Hundredth	Thousandth	Ten-thousandth	Hundred-thousandth
100,000	10,000	1,000	100	10	1	0.1	0.01	0.001	0.0001	0.00001
10^5	10^4	10^3	10^2	10^1	10^0	$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$

We usually refer to the places to the right of one's place as the decimal places.

When the number 3284.569 is written in decimal notation the decimal point locates the one's place. The number is read, "Three thousand two hundred eighty-four and five hundred sixty-nine thousandths" or "Three two eight four--point--five six nine."

Example:

Write $5 \cdot (10^2) + 7 \cdot (10^1) + 3 \cdot (\frac{1}{10})$ in decimal notation.

Notice that the one's place was not written. Could one's place be written $0 \cdot (10^0)$? Your answer, 570.3, will help you answer this question.

Exercises 6-9a

(Class Discussion)

1. Write each of the following in decimal notation:

(a) $6 \cdot (10) + 5 \cdot (1) + 8 \cdot (\frac{1}{10}) + 7 \cdot (\frac{1}{10^2})$

(b) $4 \cdot (10^2) + 3 \cdot (10) + 6 \cdot (1) + 1 \cdot (\frac{1}{10}) + 9 \cdot (\frac{1}{10^2})$

(c) $5 \cdot (10) + 2 \cdot (\frac{1}{10}) + 4 \cdot (\frac{1}{10^2})$

(d) $4 \cdot (\frac{1}{10}) + 8 \cdot (\frac{1}{10^2}) + 3 \cdot (\frac{1}{10^3})$

$$(e) 2 \cdot \left(\frac{1}{10^3}\right) + 6 \cdot \left(\frac{1}{10^4}\right)$$

$$(f) 3 \cdot \left(\frac{1}{10}\right) + 5 \cdot \left(\frac{1}{10^5}\right)$$

$$(g) 3 \cdot (10^2) + 4 \cdot \left(\frac{1}{10^2}\right)$$

2. Write each of the following in expanded form.

$$(a) 52.55$$

$$(b) 1.213$$

$$(c) 0.4$$

$$(d) 3.01$$

$$(e) 0.0102$$

$$(f) 0.10001$$

$$(g) 30.03$$

3. Below are six fractions that name rational numbers. Use the multiplication property of 1 to rename each rational number in decimal notation.

$$\text{Example: } \frac{1}{8} = \frac{125}{125} \cdot \frac{1}{8} = \frac{125}{1000} = .125$$

$$(a) \frac{5}{2}$$

$$(c) \frac{3}{8}$$

$$(e) \frac{7}{25}$$

$$(b) \frac{3}{4}$$

$$(d) \frac{4}{5}$$

$$(f) \frac{9}{20}$$

6-10. Operations with Decimals

Suppose we wish to add two rational numbers in decimal form; for example, $0.73 + 0.84$. We know how to add: $73 + 84 = 157$. How do we handle the decimal places? We may proceed as follows:

$$0.73 = 73 \cdot \frac{1}{100} \quad \text{and} \quad 0.84 = 84 \cdot \frac{1}{100}.$$

Therefore,

$$\begin{aligned} 0.73 + 0.84 &= (73 \cdot \frac{1}{100}) + (84 \cdot \frac{1}{100}) \\ &= (73 + 84) \cdot \frac{1}{100} \\ &= 157 \cdot \frac{1}{100} = 1.57. \end{aligned}$$

Notice that we have used the distributive property here.

Suppose now we wish to find the sum, $0.73 + 0.125$. Just as before, we first rename our numbers as fractions:

$$0.73 = 73 \cdot \frac{1}{100} = 730 \cdot \frac{1}{1000} \quad \text{and} \quad 0.125 = 125 \cdot \frac{1}{1000}.$$

We use the last form of 0.73 , since then $\frac{1}{1000}$ appears in both products.

$$\begin{aligned} 0.73 + 0.125 &= (730 \cdot \frac{1}{1000}) + (125 \cdot \frac{1}{1000}) \\ &= (730 + 125) \cdot \frac{1}{1000} \\ &= 855 \cdot \frac{1}{1000} = 0.855. \end{aligned}$$

These examples can be handled more conveniently by writing one number below the other as follows:

$$\begin{array}{r} .73 \\ + .84 \\ \hline 1.57 \end{array}$$

$$\begin{array}{r} 0.73 \\ + 0.125 \\ \hline 0.855 \end{array}$$

Notice that we write the decimal points directly under one another. This is because we want to add the number in the $\frac{1}{10}$ place in the first addend to the number in the $\frac{1}{10}$ place in the second addend, and the number in the $\frac{1}{100}$ place in the first addend to the number in the $\frac{1}{100}$ place in the second, etc. Thus,

$$0.73 = \frac{7}{10} + \frac{3}{100} \quad \text{and} \quad 0.84 = \frac{8}{10} + \frac{4}{100}$$

and therefore

$$\begin{aligned}0.73 + 0.84 &= \frac{7+8}{10} + \frac{3+4}{100} \\&= \frac{15}{10} + \frac{7}{100} \\&= 1.57\end{aligned}$$

Exercises 6-10a

1. Add the following numbers.
(a) $0.76 + 0.84$ (c) $1.002 + 0.0102$
(b) $0.719 + 0.382$ (d) $1.05 + 0.75 + 21.5$
2. Add the following.
(a) $0.84 + 0.76$ (c) $0.500 + 0.125$
(b) $0.625 + 0.550$ (d) $1.005 + 0.0005$
3. Perform the indicated additions.
(a) $1.051 + 0.702 + 0.066$
(b) $0.407 + 0.32 + 0.076$
4. Four men enter a hardware store, and the first wants to buy 10.1 feet of copper wire, the second wants 15.1 feet, the third wants 8.6 feet, and the fourth wants 16.6 feet. The storekeeper has 50 feet of wire in his store. Can he give each man what he wants?
5. A storekeeper has 11.5 pounds of sugar. A woman buys 5.6 pounds. Another woman buys 4.8 pounds. Then a delivery truck brings 25 pounds to the store. Finally, mice eat 0.05 pounds. How much sugar is left?
6. There are 16 ounces in 1 pound. Which is heavier, 7 ounces or 0.45 pounds?
7. In most of Europe, distances are measured in kilometers (a kilometer is about $\frac{5}{8}$ of a mile). The distance from city A to Paris is 37.5 kilometers and the distance from city B to Paris is 113.2 kilometers. How far in kilometers is it from city A to city B by way of Paris?

8. Suppose the three cities, A, B and Paris in the previous problem are on the same road, and city A is between Paris and city B. Then how far in kilometers is it between cities A and B along this road?

Suppose we wish to multiply two numbers in decimal form. For example, $0.3 \cdot 0.25$. We know how to multiply these numbers: $3 \cdot 25 = 75$. Just as before, we write

$$\begin{aligned} 0.3 \cdot 0.25 &= 3 \cdot \frac{1}{10} \cdot 25 \cdot \frac{1}{100} \\ &= 3 \cdot 25 \cdot \frac{1}{10} \cdot \frac{1}{100} \\ &= 75 \cdot \frac{1}{1000} \\ &= 0.075 \end{aligned}$$

- (1) How many digits are there to the right of the decimal point in 0.3?
- (2) How many digits are there to the right of the decimal point in 0.25?
- (3) What is the sum of the answers to (1) and (2)?
- (4) How many digits are there to the right of the decimal point in 0.075?
- (5) Compare the answers to (3) and (4).

Now multiply $0.4 \cdot 0.25$. What is your answer? Answer the five questions above, (1), (2), (3), (4), (5) for these numbers. Do the answers to (3) and (4) still agree?

To find the number of decimal places in the product when two numbers are multiplied, add the number of decimal places in the two numerals.

For example, suppose we wish to multiply 0.732 by 0.25. The first numeral has three decimal places and the second has two, so there will be five decimal places in the answer. We find the product $732 \cdot 25$, and then mark off five decimal places in the answer, counting from right to left.

$$\begin{array}{r}
 .732 \\
 .25 \\
 \hline
 3660 \\
 1464 \\
 \hline
 .18300
 \end{array}$$

We consider now the problem of dividing one number in decimal form by another, for example, $0.125 \div 0.5$.

In this case, we start with $\frac{0.125}{0.5}$. Since $\frac{a}{b} = a \cdot \frac{1}{b}$ ($b \neq 0$) then

$$\begin{aligned}
 \frac{0.125}{0.5} &= 0.125 \cdot \left(\frac{1}{0.5}\right) \\
 &= 0.125 \cdot \left(\frac{1}{\frac{1}{2}}\right) \\
 &= 0.125 \cdot (2) \\
 &= 0.250
 \end{aligned}$$

We could just as easily have used the multiplication property of 1 to arrive at the same result.

$$\begin{aligned}
 \frac{0.125}{0.5} &= \frac{0.125}{0.5} \cdot \frac{10}{10} \\
 &= \frac{1.250}{5} \\
 &= 0.250
 \end{aligned}$$

Try a more complicated fraction:

$$\begin{aligned}
 \frac{5.313}{2.53} &= 5.313 \cdot \left(\frac{1}{2.53}\right) \\
 &= 5.313 \cdot \left(\frac{1}{\frac{253}{100}}\right) \\
 &= 5.313 \cdot \left(\frac{100}{253}\right) \\
 &= \frac{531.3}{253} \\
 &= 2.1
 \end{aligned}$$

Again, we could simplify the process by using our multiplication property of 1. For that matter, by using 1 named as $\frac{1000}{1000}$ we eliminate decimal places altogether.

$$\frac{5.313}{2.53} = \frac{5.313}{2.53} \cdot \frac{1000}{1000}$$

$$= \frac{5313}{2530},$$

which is

a perfectly acceptable rational number. If you wish to name this number differently, your knowledge of the division algorithm will enable you to do so.

Exercises 6-10b

(Assume that the decimals are exact.)

1. Find the following products.

(a) $0.009 \cdot 0.09$

(b) $0.0025 \cdot 2.5$

(c) $1.2 \cdot 120$

(d) $0.135 \cdot 0.202$

2. Find answers to the following problems.

(a) $0.009 \div 30$

(b) $0.015 \div 0.05$

(c) $0.575 \div 0.4$

(d) $2.04 \div 0.008$

3. Express the following numbers as decimals.

(a) $\frac{3000}{8}$

(d) $\frac{3}{8}$

(b) $\frac{300}{8}$

(e) $\frac{3}{80}$

(c) $\frac{30}{8}$

(f) $\frac{3}{800}$

4. $\frac{.015 \cdot .0025 \cdot 2.5}{.05 \cdot .03} = ?$

6-11. The Trichotomy and Transitive Properties

We know that

$$a < b$$

means that on the number line a lies to the left of b . In the same manner we know that

$$a > b$$

means that on the number line a lies to the right of b .

From this bit of knowledge we can now state one of the most obvious properties of the order relations, $<$ and $>$. This simple property is called by the rather terrifying name, trichotomy, which means literally "cut into three cases".

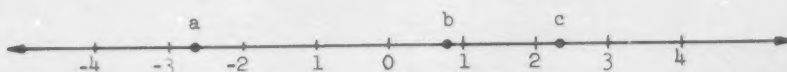
Trichotomy Property: If a and b are rational numbers, then exactly one of the following is true:

$$(i) \ a < b$$

$$(ii) \ a > b$$

$$(iii) \ a = b$$

Another easily seen property of order is contained in the observation that if a is to the left of b and b is to the left of c , then a is to the left of c .



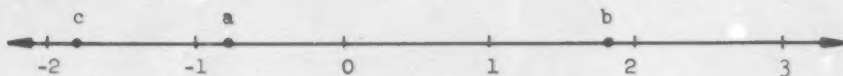
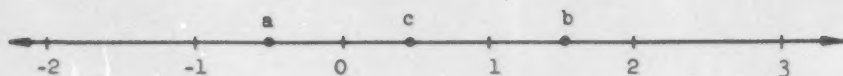
Expressing this fact in terms of the order relation, $<$, we have the property:

Transitive Property: If $a < b$ and $b < c$, then $a < c$.

There is also a transitive property for the relation $>$ as follows:

If $a > b$ and $b > c$, then $a > c$.

We must be very careful in using the transitive property. For if $a < b$ and $c < b$ we will only know that a and c are both less than b . But we will not be able to make any conclusion concerning the order of a and c , as is shown in the following two drawings.



Exercises 6-11

1. Fill in the table below. Problems (a) and (b) are done for you.

	a	b	Order of a and b	Value of $b - a$
(a)	12	15	$a < b$	$15 - 12 = 3$
(b)	$(-\frac{2}{7})$	$(-\frac{5}{7})$	$a > b$	$(-\frac{5}{7}) - (-\frac{2}{7}) = (-\frac{3}{7})$
(c)	3	5		
(d)	(-3)	5		
(e)	$(-\frac{4}{3})$	$\frac{7}{5}$		
(f)	$\frac{4}{3}$	$\frac{7}{5}$		
(g)	$(-\frac{4}{3})$	$(-\frac{7}{5})$		
(h)	$\frac{4}{3}$	$(-\frac{7}{5})$		
(i)	$(-2\frac{1}{2})$	$(-\frac{5}{2})$		
(j)	$(-\frac{7}{9})$	$(-\frac{3}{4})$		

2. Study the chart of Problem 1; then answer the following.

- (a) If $a < b$ then $(b - a)$ will be a _____?_____ number.
 (b) If $a > b$ then $(b - a)$ will be a _____?_____ number.
 (c) If $a = b$ then $(b - a)$ will be the number _____?_____.

3. In each part express any statements using ">" in terms of "<". Then see whether the transitive property can be applied to determine that $a < c$ or that $c < a$. If no conclusion is possible concerning the order of a and c write, "cannot tell".

- | | |
|-------------------------|-------------------------|
| (a) $a < b$ and $b < c$ | (f) $b > a$ and $c < b$ |
| (b) $a > b$ and $b > c$ | (g) $b < a$ and $c < b$ |
| (c) $a < b$ and $c > b$ | (h) $a < b$ and $c < b$ |
| (d) $a > b$ and $b < c$ | (i) $b > a$ and $c > b$ |
| (e) $a > b$ and $c < b$ | |

6-12. Another Look at Structure

From our work in this chapter it is obvious that the properties discovered in our work with integers still holds for rational numbers. Furthermore, we have found a property which corresponds to the addition property of opposites, namely the multiplication property of reciprocals. Our table is now complete.

Notice that there are five corresponding properties for each operation and a single property (the distributive property) that connects the two.

The Rational Number Field

a, b, and c represent rational numbers	
Closure Property of Addition $a + b$ is a rational number	Closure Property of Multiplication $a \cdot b$ is a rational number
Commutative Property of Addition $a + b = b + a$	Commutative Property of Multiplication $a \cdot b = b \cdot a$
Associative Property of Addition $(a + b) + c = a + (b + c)$	Associative Property of Multiplication $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Addition Property of Zero $a + 0 = a$	Multiplication Property of One $1 \cdot a = a$
Addition Property of Opposites $a + (-a) = 0$	Multiplication Property of Reciprocals $a \cdot \frac{1}{a} = 1 \quad (a \neq 0)$
Distributive Property of Multiplication Over Addition $a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{or}$ $(a \cdot b) + (a \cdot c) = a \cdot (b + c)$	

Exercise 6-12

1. Make a copy of the table below putting a check (✓) in each box where the indicated number system has the indicated property.

PROPERTY					
		Addition Property of Zero	Addition Property of Opposites	Multiplication Property of One	Multiplication Property of Reciprocals
	Positive Integers				
	Nonnegative Integers				
	Integers				
	Positive Rationals				
	Nonnegative Rationals				
	Rationals				

6-13. Summary

Section 6-2.

Because of the need to have closure under "division" we extend the number system to include the set of rational numbers.

Between any two distinct points, no matter how close, there are infinitely many points.

Rational numbers are frequently called "fractions". We mean by a "fraction" a symbol which indicates the quotient of two integers; division by zero is excluded. A number which can be (not must be) represented by a fraction indicating the quotient of two integers (excluding division by zero) is called a rational number.

Section 6-3.

Using symbols, any number of the form $\frac{a}{b}$, $b \neq 0$, a and b integers, is a rational number.

$$\frac{a}{b} \text{ can be expressed as } a \cdot \frac{1}{b}.$$

Therefore, $\frac{a}{b}$ can be thought of as an indicated division and also an indicated product.

Section 6-4.

Two numbers are said to be reciprocals of each other if their product is 1.

If $a \neq 0$, $\frac{a}{a} = a \cdot \frac{1}{a}$, which is read a times the reciprocal of a .

Every rational number (except zero) has a reciprocal.

Section 6-5.

If $b \neq 0$, then

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b} \text{ and } \frac{a}{b} = \frac{-a}{-b}.$$

Section 6-6.

To multiply two rational numbers named as fractions, find the fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators.

In symbols, if $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers,

$$(1) \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}.$$

Also, the process can be reversed:

$$(2) \frac{ac}{bd} = \frac{a \cdot c}{b \cdot d} = \frac{a}{b} \cdot \frac{c}{d}.$$

A prime number is an integer which has exactly 2 different integral factors. With the exception of zero and one, all integers which are not prime are called composite numbers.

All composite numbers can be expressed as a product of prime factors.

Expressing numerator and denominator of a fraction as the product of prime factors enables us to find the "simplest" name for a rational number.

Section 6-7.

The multiplication property of 1 and the distributive property provide the means for adding rational numbers in fraction form.

In symbols, if $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers,

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{bd}{bd} \cdot \left(\frac{a}{b} + \frac{c}{d} \right) = \frac{1}{bd} \cdot bd \left(\frac{a}{b} + \frac{c}{d} \right) \\ &= \frac{1}{bd} \cdot \left(\frac{bda}{b} + \frac{bdc}{d} \right) \\ &= \frac{1}{bd} \cdot (ad + bc) \\ &= \frac{ad + bc}{bd} \end{aligned}$$

Section 6-8.

A ratio can be used to compare two numbers.

The ratio of a number c to a number d , $d \neq 0$, is $\frac{c}{d}$.

The symbol % is used for the word "percent". This symbol is a short way of saying $\frac{1}{100}$. Any rational number $\frac{a}{b}$ can be expressed as a percent by finding the number c such that

$$\frac{a}{b} = \frac{c}{100} = c \cdot \frac{1}{100} = c\%.$$

Section 6-9.

A numeral such as 3284 is said to be in positional notation. In base ten this form is also called decimal notation.

The decimal point locates the one's place. We usually refer to the places to the right of the one's place as the decimal places.

Section 6-10.

The properties of addition and multiplication make it possible to operate with decimal numerals as if they named integers.

Section 6-11.

Trichotomy Property: If a and b are rational numbers, then exactly one of the following is true:

$$a < b,$$

$$a > b,$$

$$a = b.$$

Transitive Property: If $a < b$ and $b < c$, then $a < c$.

Section 6-12.

With the inclusion of the multiplication property of reciprocals, the table of field properties is completed.